CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Babak Ayazifar and Satish Rao

HW 12

Due: Friday, April 26, 2019 at 10 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 A Meeting of Three

James, Jon, and Mike each arrive at the movie theater at times uniformly distributed in the interval $(3{:}00, 3{:}20)$. Their arrival times are independent. Each person waits 5 minutes after their arrival before heading into the theater. What is the probability they all see each other before going into the theater?

2 Exponential Practice

(a) Let $X_1, X_2 \sim \text{Exponential}(\lambda)$ be independent, $\lambda > 0$. Calculate the density of $Y := X_1 + X_2$. [*Hint*: One way to approach this problem would be to compute the CDF of Y and then differentiate the CDF.]

(b) Let $t > 0$. What is the density of X_1, conditioned on $X_1 + X_2 = t$? [*Hint*: Once again, it may be helpful to consider the CDF $\mathbb{P}(X_1 \le x \mid X_1 + X_2 = t)$. To tackle the conditioning part, try conditioning instead on the event $\{X_1 + X_2 \in [t, t + \varepsilon]\}$, where $\varepsilon > 0$ is small.]

3 Normal Darts?

Alex and John are playing a game of darts. Let (X_a, Y_a) and (X_j, Y_j) denote the coordinates of Alex's and John's darts on the board and are distributed in the following way:

1. $X_a, Y_a \sim \mathbb{N}(0, 1)$ independently

2. X_j, Y_j are distributed uniformly in a circle of radius 3

The winner of the game is determined by whoever's darts is closer to the center of the board at $(0,0)$. In this question, we will compute the probability that Alex wins the game. We will denote the squared distances of the darts from the center by $r_a = X_a^2 + Y_a^2$ and by $r_j = X_j^2 + Y_j^2$.

(a) What is the distribution of r_a?
Hint: Consider the joint distribution and the following change of variables formula: Suppose we want to integrate the function $f(x,y)$ over the circle $\left(\sqrt{x^2+y^2} \le R\right)$. Then, we have the following change of variables formula:

$$\int_{\sqrt{x^2+y^2} \le R} f(x,y)dxdy = \int_0^R \int_0^{2\pi} f(r\cos\theta, r\sin\theta) r d\theta dr$$

You may find the identity $\sin(\theta)^2 + \cos(\theta)^2 = 1$ useful.

(b) What is the distribution of r_j? (Hint: Try computing the CDF first)

(c) What is the probability that Alex wins the game?

4 Why Is It Gaussian?

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where a and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

5 Moments of the Gaussian

$\mathbb{E}[X^k]$, where $k \in \mathbb{N}$, is called the *kth moment* of the distribution. In this problem, we will calculate the moments of a standard normal distribution.

(a) Prove the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) dx = t^{-1/2} \tag{1}$$

for $t > 0$.
Hint: Consider a normal distribution with variance $\frac{1}{t}$ and mean 0.

(b) For the rest of the problem, X is a standard normal distribution (with mean 0 and variance 1). Use part a to compute $\mathbb{E}[X^{2k}]$ for $k \in \mathbb{N}$. [*Hint*: Try differentiating with respect to t k times.]

(c) Compute $\mathbb{E}[X^{2k+1}]$ for $k \in \mathbb{N}$.

6 Noisy Love

Suppose you have confessed to your love interest on Valentine's Day and you are waiting to hear back. Your love interest is trying to send you a binary message: "0" means that your love interest is not interested in you, while "1" means that your love interest reciprocates your feelings. Let X be your love interest's message for you. Your current best guess of X has $\mathbb{P}(X = 0) = 0.7$ and $\mathbb{P}(X = 1) = 0.3$. Unfortunately, your love interest sends you the message through a noisy channel, and instead of receiving the message X, you receive the message $Y = X + \varepsilon$, where ε is independent Gaussian noise with mean 0 and variance 0.49.

(a) First, you decide upon the following rule: if you observe $Y > 0.5$, then you will assume that your love interest loves you back, whereas if you observe $Y \le 0.5$, then you will assume that your love interest is not interested in you. What is the probability that you are correct using this rule? (Express your answer in terms of the CDF of the standard Gaussian distribution $\Phi(z) = \mathbb{P}(\mathcal{N}(0,1) \le z)$, and then evaluate your answer numerically.)

(b) Suppose you observe $Y = 0.6$. What is the probability that your love interest loves you back? [*Hint*: This problem requires conditioning on an event of probability 0, namely, the event $\{Y = 0.6\}$. To tackle this problem, think about conditioning on the event $\{Y \in [0.6, 0.6+\delta]\}$, where $\delta > 0$ is small, so that $f_Y(0.6) \cdot \delta \approx \mathbb{P}(Y \in [0.6, 0.6+\delta])$, and then apply Bayes Rule.]

(c) Suppose you observe $Y = y$. For what values is it more likely than not that your love interest loves you back? [*Hint*: As before, instead of considering $\{Y = y\}$, you can consider the event $\{Y \in [y, y+\delta]\}$ for small $\delta > 0$. So, when is $\mathbb{P}(X = 1 \mid Y \in [y, y+\delta]) \ge \mathbb{P}(X = 0 \mid Y \in [y, y+\delta])$?]

(d) Your new rule is to assume that your love interest loves you back if (based on the value of Y that you observe) it is more likely than not that your love interest loves you back. Under this new rule, what is the probability that you are correct?