CS 70 Discrete Mathematics and Probability Theory
Spring 2019 Satish Rao and Babak Ayazifar **Homework 7**

Due: Friday, March 15, 2019 at 10 PM

Sundry

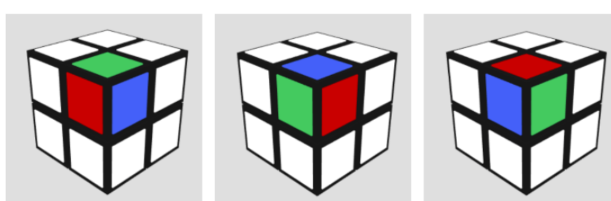
Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Rubik's Cube Scrambles

We wish to count the number of ways to scramble a $2\times2\times2$ Rubik's Cube, and take a quick look at the $3\times3\times3$ cube. Leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) The $2\times2\times2$ Rubik's Cube is composed of 8 "corner pieces" arranged in a $2\times2\times2$ cube. How many ways can we assign all the corner pieces a position?

(b) Each corner piece has three distinct colors on it, and so can also be oriented three different ways once it is assigned a position (see figure below). How many ways can we *assemble* (assign each piece a position and orientation) a $2\times2\times2$ Rubik's Cube?



Three orientations of a corner piece

(c) The previous part assumed we can take apart pieces and assemble them as we wish. But certain configurations are unreachable if we restrict ourselves to just turning the sides of the cube. What this means for us is that if the orientations of 7 out of 8 of the corner pieces are determined, there is only 1 valid orientation for the eighth piece. Given this, how many ways are there to *scramble* (as opposed to *assemble*) a $2\times2\times2$ Rubik's Cube?

(d) We decide to treat scrambles that differ only in overall positioning (e.g. flipped upside-down or rotated but otherwise unaltered) as the same scramble. Then we overcounted in the previous part! How does this new condition change your answer to the previous part?

(e) Now consider the $3 \times 3 \times 3$ Rubik's Cube. In addition to 8 corner pieces, we now have 12 "edge" pieces, each of which can take 2 orientations. How many ways can we *assemble* a $3 \times 3 \times 3$ Rubik's Cube?

2 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. For this problem, you do not need to show work that justifies your answers. We encourage you to leave your answer as an expression (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
How many different 13-card bridge hands are there? How many different 13-card bridge hands are there that contain no aces? How many different 13-card bridge hands are there that contain all four aces? How many different 13-card bridge hands are there that contain exactly 6 spades?

(c) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(d) How many 99-bit strings are there that contain more ones than zeros?

(e) An anagram of FLORIDA is any re-ordering of the letters of FLORIDA, i.e., any string made up of the letters F, L, O, R, I, D, and A, in any order. The anagram does not have to be an English word.
How many different anagrams of FLORIDA are there? How many different anagrams of ALASKA are there? How many different anagrams of ALABAMA are there? How many different anagrams of MONTANA are there?

(f) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(g) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(h) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student?

(k) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(l) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(m) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

3 Divisor Graph Colorings

Define G where we have $V = \{2,3,4,5,6,7,8,9\}$, and we add an edge between vertex i and vertex j if i divides j, or j divides i.

(a) Explain why we cannot vertex-color G with only 2 colors.

(b) How many ways can we vertex-color G with 3 colors?

4 Vacation Time

After a number of complaints, the Dunder Mifflin Paper Company has decided on the following rule for vacation leave for the next year (365 days): Every employee must take exactly one vacation leave of 4 consecutive days, one vacation leave of 5 consecutive days and one vacation leave of 6 consecutive days within the year, with the property that any two of the vacation leaves have a gap of at least 7 days between them. In how many ways can an employee arrange their vacation time? (The vacation policy resets every year, so there is no need to worry about leaving a gap between this year and next year's vacations).

5 Story Problems

Prove the following identities by combinatorial argument:

(a) $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) $n^2 = 2\binom{n}{2} + n$

(c) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
Hint: Consider how many ways there are to pick groups of people ("teams") and then a representative ("team leaders").

(d) $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$
Hint: Consider a generalization of the previous part.

6 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.