CS 70 Spring 2019 | Discrete Mathematics and Probability Theory
Satish Rao and Babak Ayazifar

HW 3

Due: Februrary 15, 2019 at 10 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Leaves in a Tree

A *leaf* in a tree is a vertex with degree 1.

(a) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) Prove that every tree on $n \geq 2$ vertices must have at least two leaves.

2 Coloring Trees

(a) What is the minimum number of colors needed to color a tree? Prove your claim.

(b) Prove that all trees are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* How does your answer to part (a) relate to this?]

3 Edge-Disjoint Paths in a Hypercube

Prove that between any two distinct vertices x, y in the n-dimensional hypercube graph, there are at least n edge-disjoint paths from x to y (i.e., no two paths share an edge, though they may share vertices).

4 Triangulated Planar Graph

In this problem you will prove that every triangulated planar graph (every face has 3 sides; that is, every face has three edges bordering it, including the unbounded face) contains either (1) a vertex of degree 1, 2, 3, 4, (2) two degree 5 vertices which are adjacent, or (3) a degree 5 and a degree 6 vertices which are adjacent. Justify your answers.

(a) Place a "charge" on each vertex v of value $6 - \text{degree}(v)$. What is the sum of the charges on all the vertices? (*Hint*: Use Euler's formula and the fact that the planar graph is triangulated.)

(b) What is the charge of a degree 5 vertex and of a degree 6 vertex?

(c) Suppose now that we shift $1/5$ of the charge of a degree 5 vertex to each of its neighbors that has a negative charge. (We refer to this as "discharging" the degree 5 vertex.) Conclude the proof under the assumption that, after discharging all degree 5 vertices, there is a degree 5 vertex with positive remaining charge.

(d) If no degree 5 vertices have positive charge after discharging the degree 5 vertices, does there exist any vertex with positive charge after discharging? If there is such a vertex, what are the possible degrees of that vertex?

(e) Suppose there exists a degree 7 vertex with positive charge after discharging the degree 5 vertices. How many neighbors of degree 5 might it have?

(f) Continuing from Part (e). Since the graph is triangulated, are two of these degree 5 vertices adjacent?

(g) Finish the proof from the facts you obtained from the previous parts.

5 Euclid's Algorithm

(a) Use Euclid's algorithm from lecture to compute the greatest common divisor of 527 and 323. List the values of x and y of all recursive calls.

(b) Use extended Euclid's algorithm from lecture to compute the multiplicative inverse of 5 mod 27. List the values of x and y and the returned values of all recursive calls.

(c) Find $x \pmod{27}$ if $5x + 26 \equiv 3 \pmod{27}$. You can use the result computed in (b).

(d) Assume a, b, and c are integers and $c > 0$. Prove or disprove: If a has no multiplicative inverse mod c, then $ax \equiv b \pmod{c}$ has no solution.

6 Fibonacci GCD

The Fibonacci sequence is given by $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$ and $F_1 = 1$. Prove that, for all $n \geq 0$, $\gcd(F_n, F_{n-1}) = 1$,