1 The Count

(a) How many permutations of COSTUME contain "COME" as a substring? How about as a subsequence (meaning the letters of "COME" have to appear in that order, but not necessarily next to each other)?

(b) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(c) How many ways are there to choose five nonnegative integers x_0, x_1, x_2, x_3, x_4 such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod{5}$?

(d) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

2 Charming Star

At the end of each day, students will vote for the most charming student. There are 5 candidates and 100 voters. Each voter can only vote once, and all of their votes weigh the same. A "voting combination" is defined by how many votes each candidate receives. In this question, only the number of votes for each candidate matters; it does not matter which specific people voted for each candidate.

(a) How many possible voting combinations are there for the 5 candidates?

(b) How many possible voting combinations are there such that exactly one candidate gets more than 50 votes?

3 Finicky Bins

If a "finicky" bin has at least 5 balls in it, the 5 balls will fall out and not be counted (e.g., 6 balls in a finicky bin is the same as 1). Suppose we throw 7 indistinguishable balls into 4 finicky bins. How many possible outcomes are there? We consider two outcomes to be the same if they result in the same final distribution of balls in the bins.

4 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

(b) $\binom{n}{i} = \binom{n}{n-i}$.

(c) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$.