1 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n+1$.

(b) For all positive integers n, prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) For all positive natural numbers n, $\frac{5}{4} \cdot 8^n + 3^{3n-1}$ is divisible by 19.

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

3 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.