Today

Finish Euclid.

Bijection/CRT/Isomorphism.

Fermat's Little Theorem.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Extend euclid to find inverse.

Euclid's GCD algorithm.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Computes the gcd(x,y) in $O(n)$ divisions. (Remember $n = \log_2 x$.)

For x and m, if gcd$(x,m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?

Extended GCD

Euclid's Extended GCD Theorem: For any x, y there are integers a, b such that

$ax + by = d$ where $d = gcd(x,y)$.

"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when gcd$(x,m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!

Example: For $x = 12$ and $y = 35$, gcd$(12,35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.

The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
```

Claim: Returns (d,a,b): $d = gcd(x,y)$ and $d = ax + by$.

Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y)  * b)
```

Theorem: Returns (d, a, b), where $d = gcd(x, y)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod (x, y))$ returns (d, a, b) with
$d = ay + b(\mod (x, y))$

ext-gcd(x, y) calls **ext-gcd**$(y, \mod (x, y))$ so

$$\begin{aligned} d &= ay + b \cdot (\mod (x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! □

[1] Assume d is $gcd(x, y)$ by previous proof.

Review Proof: step.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.
egcd(7,60).

$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \\ 7(9) + 60(-1) &= 3 \\ 7(-17) + 60(2) &= 1 \end{aligned}$$

Confirm: $-119 + 120 = 1$

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...
$2^{n/2}$

Inverse of 500,000,357 modulo 1,000,000,000,000?
≤ 80 divisions.
versus 1,000,000

Internet Security.
Public Key Cryptography: 512 digits.
512 divisions vs.
$(10000000000000000000000000000000000000000)^5$ divisions.

More Number Theory Tools.

Chinese remainder theorem.

Fermat's Theorem.

Bijections

Bijection is **one to one** and **onto.**
Bijection:
$f : A \to B$.
Domain: A, Co-Domain: B.
Versus Range.
E.g. **sin** (x).
$A = B =$ reals.
Range is $[-1,1]$. Onto: $[-1,1]$.
Not one-to-one. **sin** $(\pi) =$ **sin** $(0) = 0$.

Range Definition always is onto.
Consider $f(x) = ax \mod m$.
$f : \{0,\ldots,m-1\} \to \{0,\ldots,m-1\}$.
Domain/Co-Domain: $\{0,\ldots,m-1\}$.

When is it a bijection?
When $gcd(a,m)$ is ....? ... 1.

Not Example: $a = 2$, $m = 4$, $f(0) = f(2) = 0 \pmod 4$.

Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3. Not $5 \pmod 7$!

If $x = 5 \pmod 7$
then x is in $\{5,12,19,26,33\}$.

Oh, only 33 is $3 \pmod 5$.

Hmmm... only one solution.

A bit slow for large values.

Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod m$ and $x = b \pmod n$ where gcd(m,n)=1.

CRT Thm: There is a unique solution $x \pmod{mn}$.
Proof:
Consider $u = n(n^{-1} \pmod m)$.
$u = 0 \pmod n$ $u = 1 \pmod m$

Consider $v = m(m^{-1} \pmod n)$.
$v = 1 \pmod n$ $v = 0 \pmod m$

Let $x = au + bv$.
$x = au + bv = a \pmod m$: $bv = 0 \pmod m$ and $au = a \pmod m$
$x = au + bv = b \pmod n$: $au = 0 \pmod n$ and $bv = b \pmod n$

Only solution? If not, two solutions, x and y.
$(x-y) \equiv 0 \pmod m$ and $(x-y) \equiv 0 \pmod n$.
$\implies (x-y)$ is multiple of m and n since gcd(m,n)=1.
$\implies x - y \geq mn \implies x,y \notin \{0,\ldots,mn-1\}$.
Thus, only one solution modulo mn. □

Fermat's Theorem: Reducing Exponents.

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod p$,
$a^{p-1} \equiv 1 \pmod p$.

Proof: Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo p since a has an inverse modulo p.
S contains representative of $\{1,\ldots,p-1\}$ modulo p.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

Each of $2, \ldots (p-1)$ has an inverse modulo p, solve to get...

$$a^{(p-1)} \equiv 1 \mod p.$$

□

Fermat and Exponent reducing.

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod p$,
$a^{p-1} \equiv 1 \pmod p$.

What is $2^{101} \pmod 7$?

Wrong: $2^{101} = 2^{7*14+3} = 2^3 \pmod 7$

Fermat: 2 is relatively prime to 7. $\implies 2^6 = 1 \pmod 7$.

Correct: $2^{101} = 2^{6*16+5} = 2^5 = 32 = 4 \pmod 7$.

For a prime modulus, we can reduce exponents modulo $p-1$!

Isomorphisms.

Bijection:

$f(x) = ax \pmod m$ if $gcd(a,m) = 1$.

Chinese Remainder Theorem: For m,n with $gcd(n,m) = 1$,
$\implies$ unique $x \pmod{mn}$ where $x = a \pmod m$ and $x = b \pmod n$.

Bijection between $(a \pmod n, b \pmod m)$ and $x \pmod m n$.

Consider $m = 5$, $n = 9$, then if $(a,b) = (3,7)$ then $x = 43 \pmod{45}$.

Consider $(a',b') = (2,4)$, then $x = 22 \pmod{45}$.

Now consider: $(a,b) + (a',b') = (0,2)$.

What is x where $x = 0 \pmod 5$ and $x = 2 \pmod 9$?

Try $43 + 22 = 65 = 20 \pmod{45}$.

Is it $0 \pmod 5$? Yes! Is it $2 \pmod 9$? Yes!

Isomorphism:
the actions $+, \times$ under $\pmod 5$, $\pmod 9$
correspond to $+, \times$ actions in $\pmod{45}$!

Public Key Cryptography.

1. Public Key Cryptography
2. RSA system
3. Warnings.

Xor

Computer Science:
1 - True
0 - False

$1 \vee 1 = 1$
$1 \vee 0 = 1$
$0 \vee 1 = 1$
$0 \vee 0 = 0$

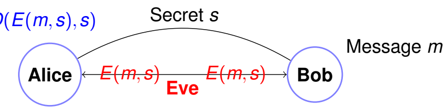
$A \oplus B$ - Exclusive or.
$1 \oplus 1 = 0$
$1 \oplus 0 = 1$
$0 \oplus 1 = 1$
$0 \oplus 0 = 0$

Note: Also modular addition modulo 2!
$\{0,1\}$ is set. Take remainder for 2.

Property: $A \oplus B \oplus B = A$.
By cases: $1 \oplus 1 \oplus 1 = 1$. ...

Cryptography ...

$m = D(E(m,s),s)$



Example:
One-time Pad: secret s is string of length $|m|$.
$m = 10101011110101101$
$s = ...$
$E(m,s)$ – bitwise $m \oplus s$.
$D(x,s)$ – bitwise $x \oplus s$.
Works because $m \oplus s \oplus s = m$!
...and totally secure!
...given $E(m,s)$ any message m is equally likely.

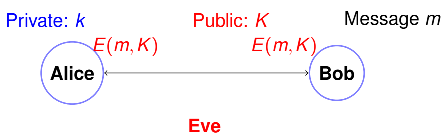
Disadvantages:

Shared secret!

Uses up one time pad..or less and less secure.

Public key crypography.

$m = D(E(m,K),k)$



Everyone knows key K!
Bob (and Eve and me and you and you ...) can encode.
Only Alice knows the secret key k for public key K.
(Only?) Alice can decode with k.

Is this even possible?

Is public key crypto possible?

We don't really know.
...but we do it every day!!!

RSA (Rivest, Shamir, and Adleman)
Pick two large primes p and q. Let $N = pq$.
Choose e relatively prime to $(p-1)(q-1)$.[2]
Compute $d = e^{-1} \mod (p-1)(q-1)$.
Announce $N(= p \cdot q)$ and e: $K = (N,e)$ is my public key!

Encoding: $\mod (x^e, N)$.

Decoding: $\mod (y^d, N)$.

Does $D(E(m)) = m^{ed} = m \mod N$?

Yes!

[2] Typically small, say $e = 3$.

Iterative Extended GCD.

Example: $p = 7$, $q = 11$.

$N = 77$.
$(p-1)(q-1) = 60$
Choose $e = 7$, since $\gcd(7,60) = 1$.
egcd(7,60).

$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \\ 7(-17)+60(2) &= 1 \end{aligned}$$

Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

Encryption/Decryption Techniques.

Public Key: $(77,7)$
Message Choices: $\{0,\ldots,76\}$.

Message: 2!

$E(2) = 2^e = 2^7 \equiv 128 \pmod{77} = 51 \pmod{77}$
$D(51) = 51^{43} \pmod{77}$
uh oh!

Obvious way: 43 multiplications. Ouch.

In general, $O(N)$ or $O(2^n)$ multiplications!

Repeated squaring.

Notice: $43 = 32+8+2+1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^8 \cdot 51^2 \cdot 51^1 \pmod{77}$.
4 multiplications sort of...
Need to compute $51^{32} \ldots 51^1$.?
$51^1 \equiv 51 \pmod{77}$
$51^2 = (51)*(51) = 2601 \equiv 60 \pmod{77}$
$51^4 = (51^2)*(51^2) = 60*60 = 3600 \equiv 58 \pmod{77}$
$51^8 = (51^4)*(51^4) = 58*58 = 3364 \equiv 53 \pmod{77}$
$51^{16} = (51^8)*(51^8) = 53*53 = 2809 \equiv 37 \pmod{77}$
$51^{32} = (51^{16})*(51^{16}) = 37*37 = 1369 \equiv 60 \pmod{77}$

5 more multiplications.

$51^{32} \cdot 51^8 \cdot 51^2 \cdot 51^1 = (60)*(53)*(60)*(51) \equiv 2 \pmod{77}$.

Decoding got the message back!

Repeated Squaring took 9 multiplications versus 43.

Recursive version.

```
(define (power x y m)
  (if (= y 1)
    (mod x m)
    (let ((x-to-evened-y  (power (square x) (/ y 2) m)))
    (if (evenp y)
        x-to-evened-y
        (mod (* x x-to-evened-y) m )))))
```

Claim: Program correctly computes x^y.

Base: $x^1 = x \pmod{m}$.

$x^y = x^{2(y/2) + \bmod (y,2)} = (x^2)^{y/2} x^{y \bmod 2} \pmod{m}$.

Last expression computed in recursive call with x^2 and $y/2$.

Note: $y/2$ is integer division.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y (in binary) is 1.
   Example: $43 = 101011$ in binary.
   $x^{43} = x^{32} * x^8 * x^2 * x^1$.

Modular Exponentiation: $x^y \bmod N$. All n-bit numbers.
Repeated Squaring:
$O(n)$ multiplications.
$O(n^2)$ time per multiplication.
$\implies O(n^3)$ time.
Conclusion: $x^y \bmod N$ takes $O(n^3)$ time.

RSA is pretty fast.

Modular Exponentiation: $x^y \bmod N$. All n-bit numbers.
$O(n^3)$ time.

Remember RSA encoding/decoding!

$E(m,(N,e)) = m^e \pmod{N}$.
$D(m,(N,d)) = m^d \pmod{N}$.

For 512 bits, a few hundred million operations.
Easy, peasey.

Decoding.

$E(m,(N,e)) = m^e \pmod{N}$.
$D(m,(N,d)) = m^d \pmod{N}$.

$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.

Want: $(m^e)^d = m^{ed} = m \pmod{N}$.

Always decode correctly?

$E(m,(N,e)) = m^e \pmod N$.
$D(m,(N,d)) = m^d \pmod N$.

$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.

Want: $(m^e)^d = m^{ed} = m \pmod N$.

Another view:
$d = e^{-1} \pmod{(p-1)(q-1)} \iff ed = k(p-1)(q-1)+1$.

Consider...

Fermat's Little Theorem: For prime p, and $a \not\equiv 0 \pmod p$,

$$a^{p-1} \equiv 1 \pmod p.$$

$\implies a^{k(p-1)} \equiv 1 \pmod p \implies a^{k(p-1)+1} = a \pmod p$

versus $a^{k(p-1)(q-1)+1} = a \pmod{pq}$.

Similar, not same, but useful.

...decoding correctness...

CRT: Isomorphism between ($a \mod p, b \mod q$) and $x \pmod{pq}$

$e = d^{-1} \mod pq$.

$x^{ed} = x^{1+k(p-1)(q-1)} \pmod{pq}$

Now $x = a \mod p$ and $x = b \pmod q$.

$a^{1+k(p-1)(q-1)} = a(a^{(p-1)})^{k(q-1)} = a \pmod p$
By Fermat. $a^{p-1} = 1 \pmod p$

$b^{1+k(p-1)(q-1)} = b(b^{(q-1)})^{k(p-1)} = b \pmod q$
By Fermat. $b^{q-1} = 1 \pmod q$

$x^{ed} = a \pmod p$ and $x^{ed} = b \pmod q$.

CRT $\implies x^{ed} = x \pmod{pq}$.

Construction of keys.. ..

1. Find large (100 digit) primes p and q?
   Prime Number Theorem: $\pi(N)$ number of primes less than N.For all $N \geq 17$
   $$\pi(N) \geq N/\ln N.$$
   Choosing randomly gives approximately $1/(\ln N)$ chance of number being a prime. (How do you tell if it is prime? ... cs170..Miller-Rabin test.. Primes in P).
   For 1024 bit number, 1 in 710 is prime.
2. Choose e with $\gcd(e,(p-1)(q-1)) = 1$.
   Use gcd algorithm to test.
3. Find inverse d of e modulo $(p-1)(q-1)$.
   Use extended gcd algorithm.

All steps are polynomial in $O(\log N)$, the number of bits.

Security of RSA.

Security?

1. Alice knows p and q.
2. Bob only knows, $N(= pq)$, and e.
   Does not know, for example, d or factorization of N.
3. I don't know how to break this scheme without factoring N.

No one I know or have heard of admits to knowing how to factor N.
Breaking in general sense $\implies$ factoring algorithm.

Much more to it.....

If Bobs sends a message (Credit Card Number) to Alice,

Eve sees it.

Eve can send credit card again!!

The protocols are built on RSA but more complicated;

For example, several rounds of challenge/response.

One trick:
Bob encodes credit card number, c,
concatenated with random k-bit number r.

Never sends just c.

Again, more work to do to get entire system.

CS161...

Signatures using RSA.

Verisign: k_v, K_v

$[C, S_v(C)]$ $C = E(S_V(C), k_V)$?

$[C, S_v(C)]$ $[C, S_v(C)]$

Amazon ⟷ **Browser.** K_v

Certificate Authority: Verisign, GoDaddy, DigiNotar,...
Verisign's key: $K_V = (N,e)$ and $k_V = d$ $(N = pq.)$
Browser "knows" Verisign's public key: K_V.
Amazon Certificate: C = "I am Amazon. My public Key is K_A."
Versign signature of C: $S_v(C)$: $D(C,k_V) = C^d \mod N$.
Browser receives: $[C, y]$
Checks $E(y, K_V) = C$?

$E(S_v(C), K_V) = (S_v(C))^e = (C^d)^e = C^{de} = C \pmod N$
Valid signature of Amazon certificate C!

Security: Eve can't forge unless she "breaks" RSA scheme.

RSA

Public Key Cryptography:

$D(E(m,K),k) = (m^e)^d \mod N = m.$

Signature scheme:

$E(D(C,k),K) = (C^d)^e \mod N = C$

Other Eve.

Get CA to certify fake certificates: Microsoft Corporation.

2001..Doh.

... and August 28, 2011 announcement.

DigiNotar Certificate issued for Microsoft!!!

How does Microsoft get a CA to issue certificate to them ...

and only them?

Summary.

Public-Key Encryption.

RSA Scheme:
$N = pq$ and $d = e^{-1} \pmod{(p-1)(q-1)}$.
$E(x) = x^e \pmod{N}$.
$D(y) = y^d \pmod{N}$.

Repeated Squaring $\implies$ efficiency.

Fermat's Theorem $\implies$ correctness.

Good for Encryption and Signature Schemes.