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The notion of "next" is undefinable.

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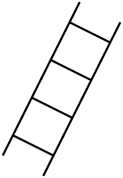

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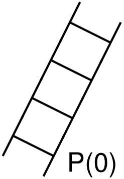

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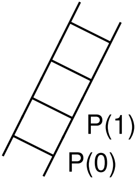

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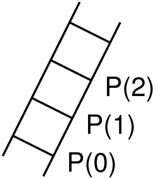

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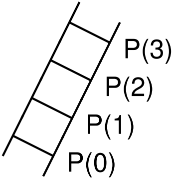

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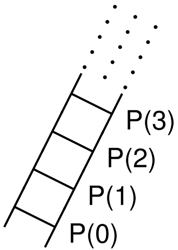

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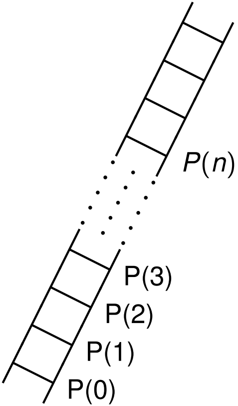

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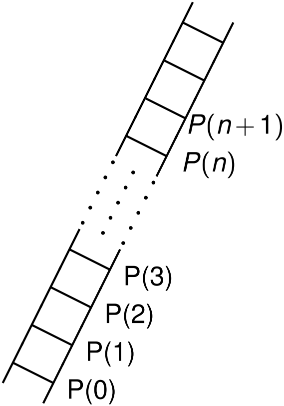

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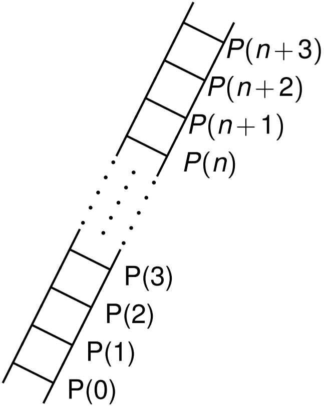

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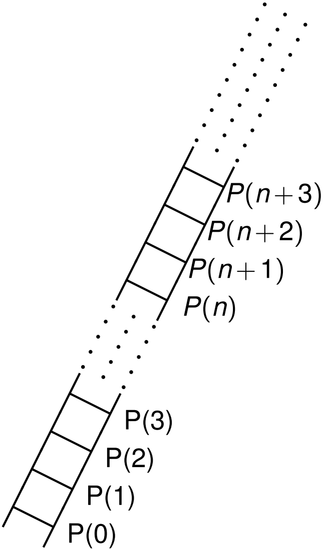

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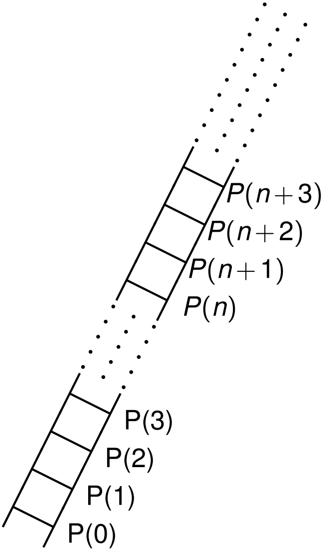

Your favorite example of forever..

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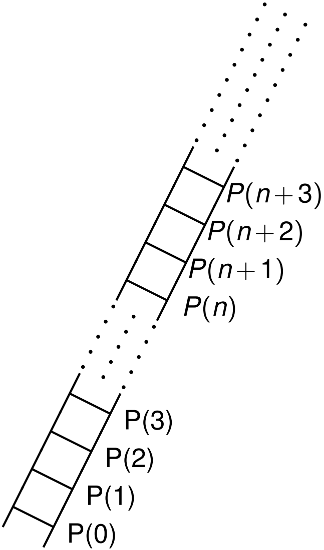

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Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$

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Idea: assume predicate $P(n)$ for $n = k$.

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Is predicate, $P(n)$ true for $n = k+1$?

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Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k + 1$?

$\sum_{i=0}^{k+1} i$

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Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

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...

true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

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Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1) = (k+1)(\frac{k}{2}+1) = \frac{(k+1)(k+2)}{2}.$$

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Predicate, $P(n)$, True for all natural numbers!

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Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

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$(k+1)^3 - (k+1)$

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$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$

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Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

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Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n))$.

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$(q + k^2 + k)$ is integer (closed under addition and multiplication).

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Thus, $(\forall k \in N) P(k) \implies P(k+1)$

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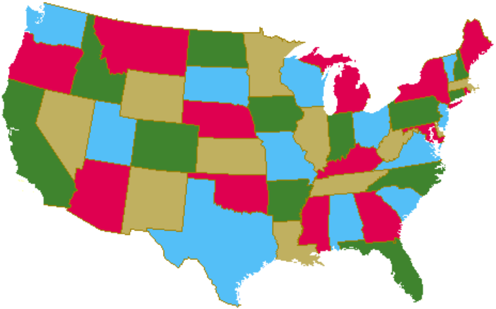
Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.



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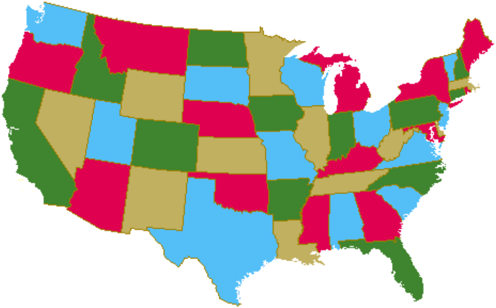

Check Out: “Four corners”.

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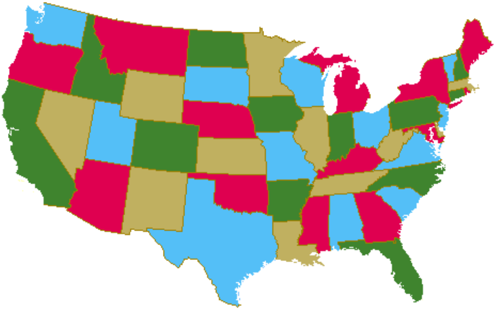

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States connected at a point, can have same color.

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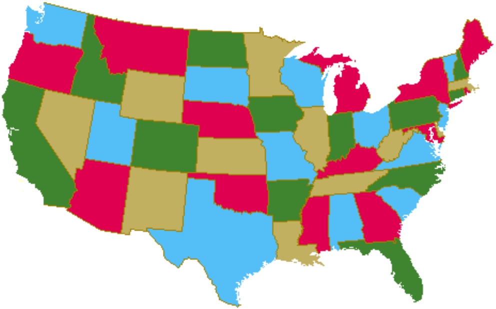

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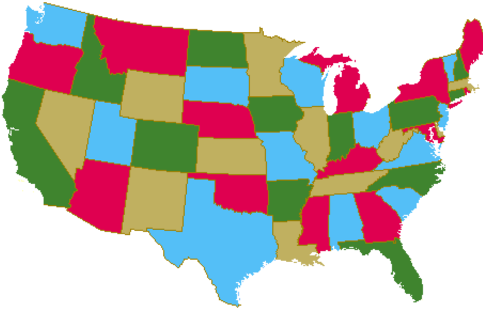

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Quick Test: Which states?

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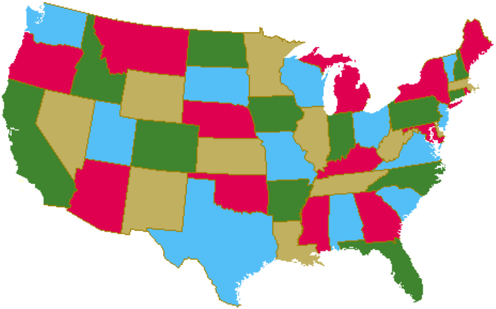

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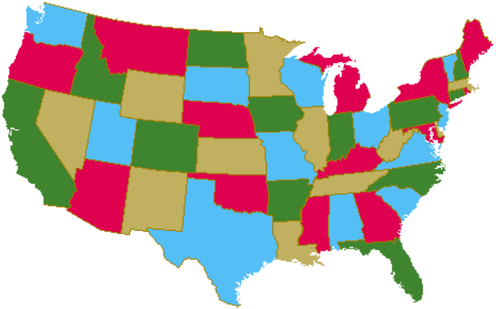

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Quick Test: Which states? Utah. Colorado.

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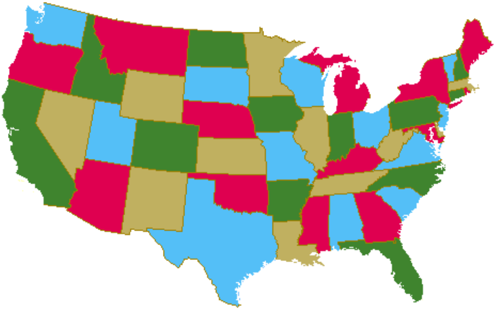

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Quick Test: Which states? Utah. Colorado. New Mexico.

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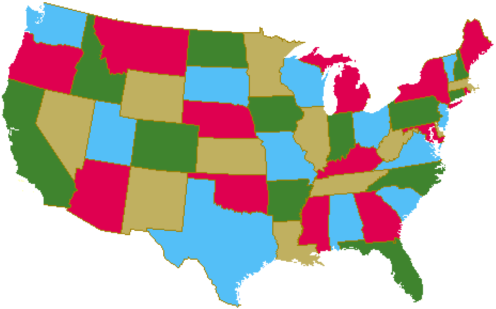

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Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

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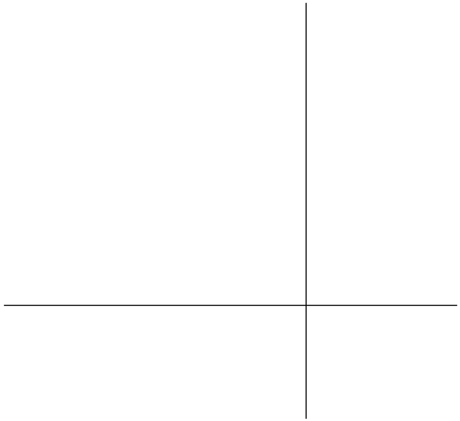

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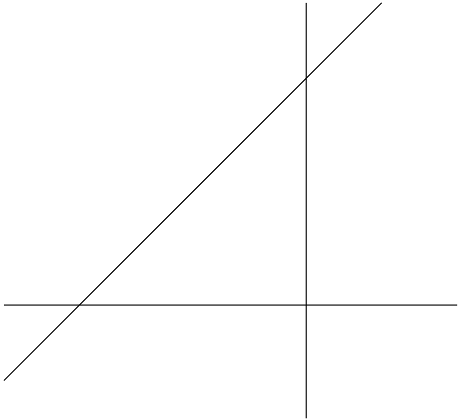

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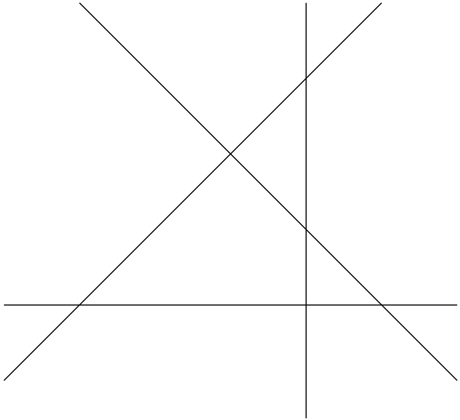
Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.



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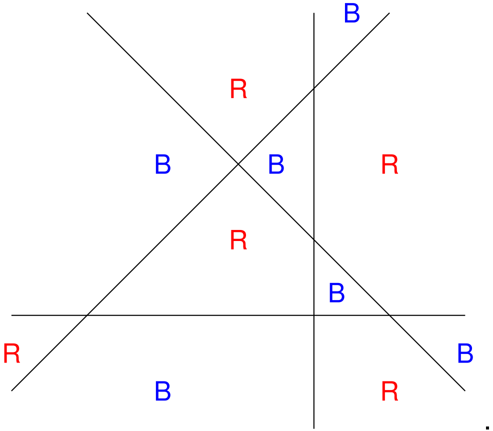

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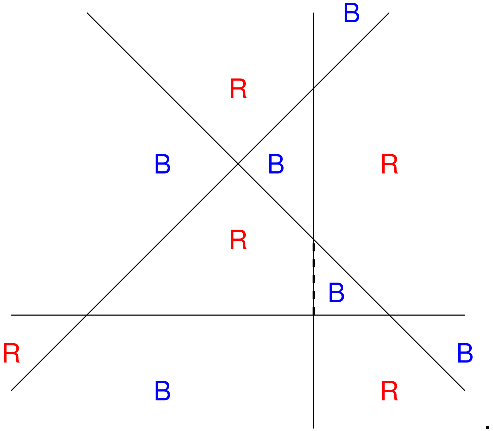

Fact: Swapping red and blue gives another valid colors.

Two color theorem: example.

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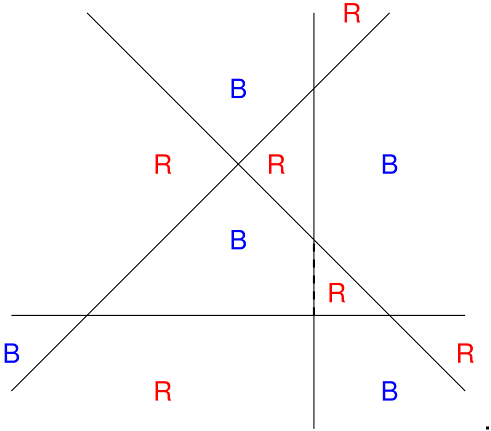

Fact: Swapping red and blue gives another valid colors.

Swapping works.

What does works mean?

Swapping works.

What does works mean?

Gives another valid coloring?

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What does works mean?

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What does valid mean?

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What does valid mean?

For each line segment or ray, any two regions of the plane that are separated by the ray or line segment are different colors.

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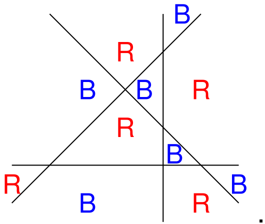

Swapping a valid coloring works.

It was valid, so adjacent regions were different colors, changing both colors to each other, implies they are still different.

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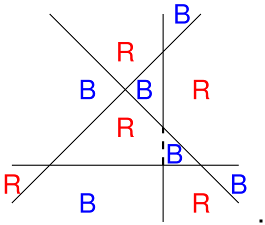

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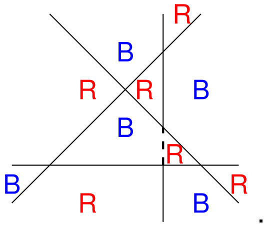

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What does works mean?

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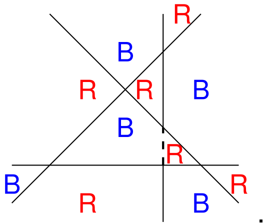

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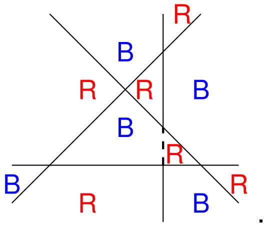

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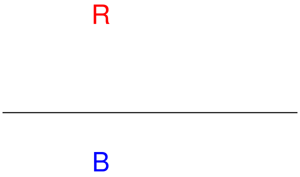
Two color theorem: proof illustration.

Base Case.

Two color theorem: proof illustration.



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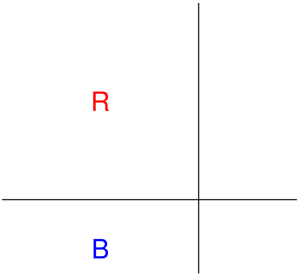
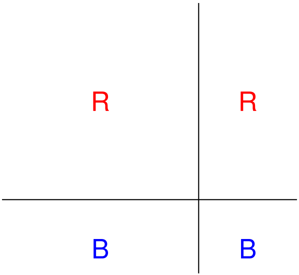


1. Add line.

Two color theorem: proof illustration.



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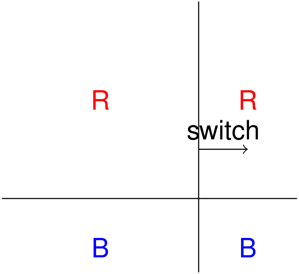

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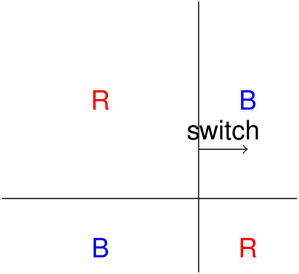
3. Switch on one side of new line.

(Fixes conflicts along line, and makes no new ones.)

Two color theorem: proof illustration.



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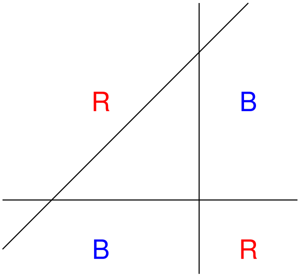

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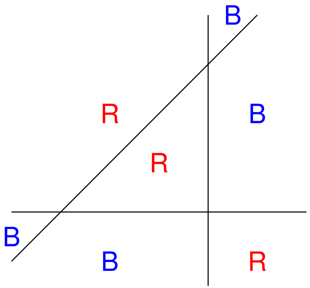

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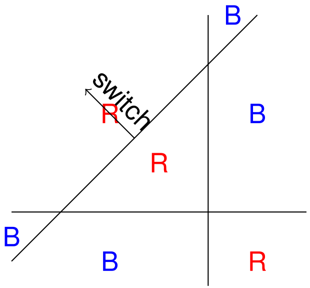

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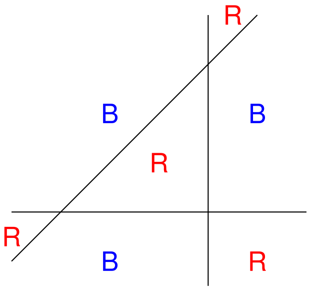

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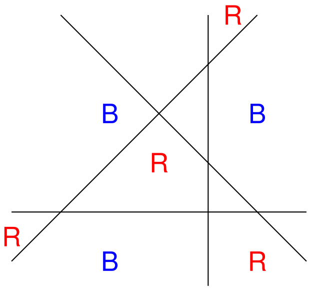

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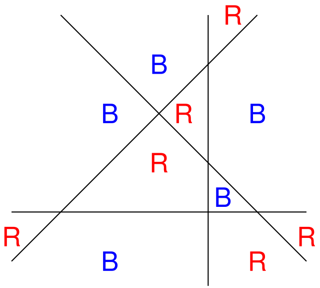

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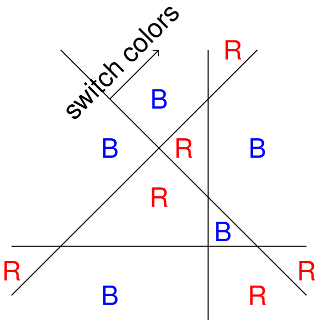

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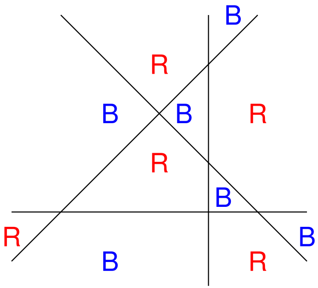

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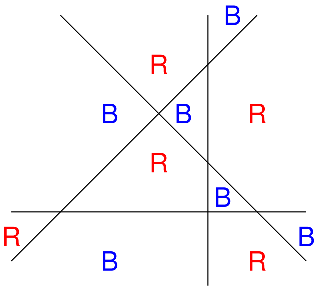

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Algorithm gives $P(k) \implies P(k+1)$.

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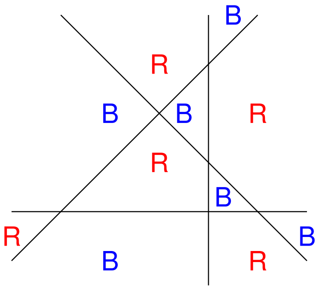

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Strenthening Induction Hypothesis.

Theorem: The sum of the first *n* odd numbers is a perfect square.

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kth odd number is $2(k-1)+1$.

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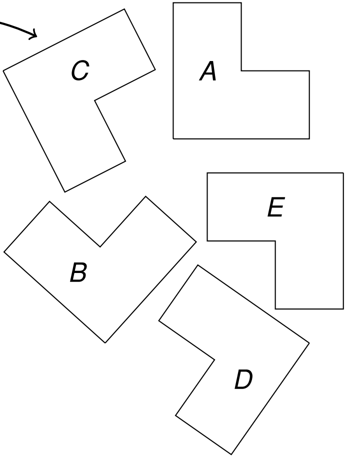
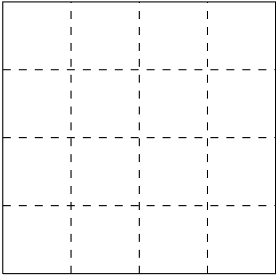
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



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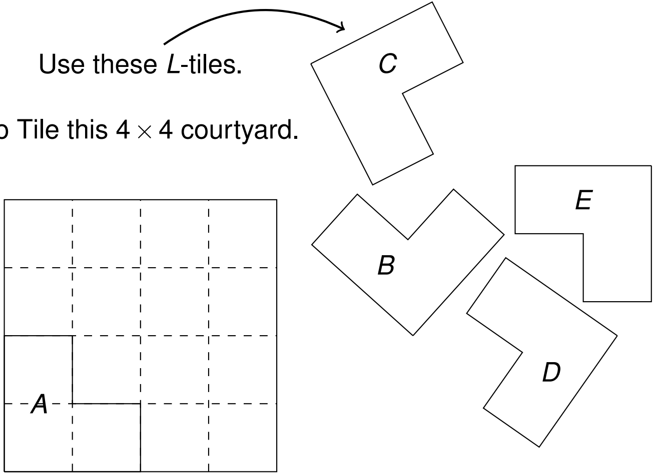

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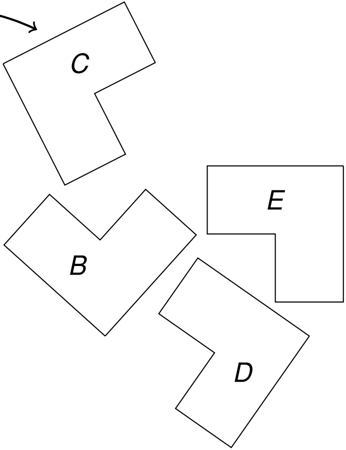

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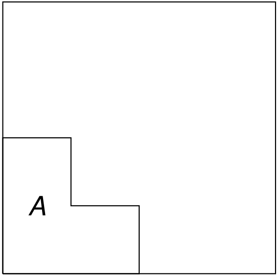

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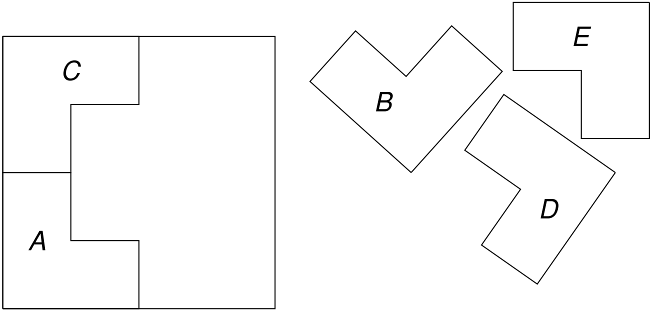

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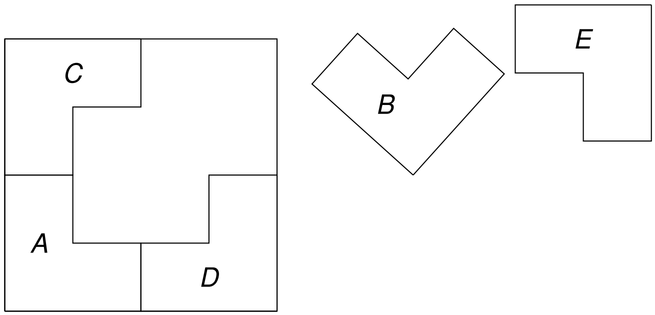

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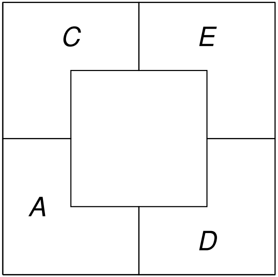

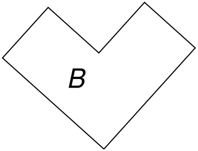

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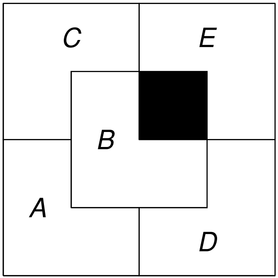

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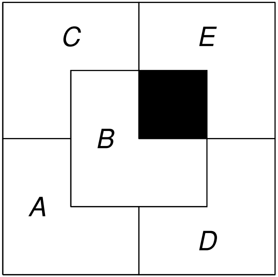

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Tiling Cory Hall Courtyard.

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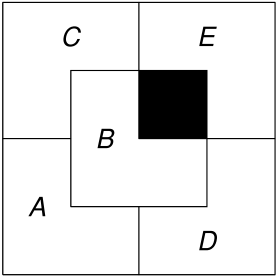


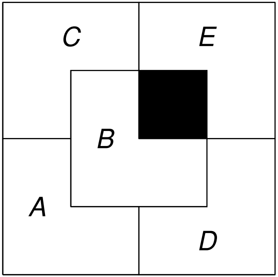
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

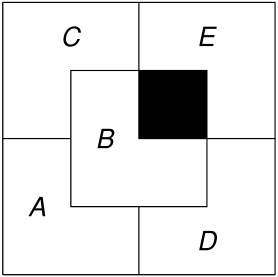
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



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Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

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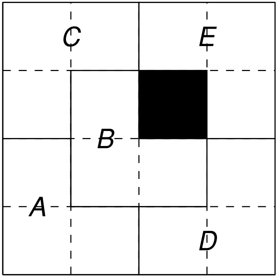

Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

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Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

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Proof: The remainder of 2^{2n} divided by 3 is 1.

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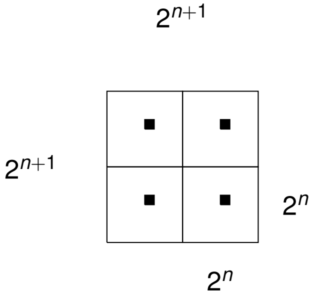

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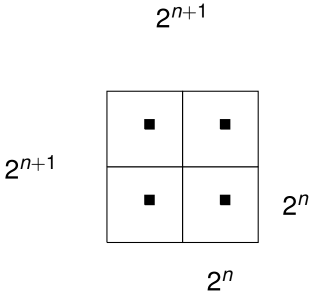

What to do now???

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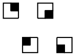

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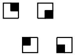

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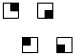

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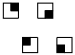

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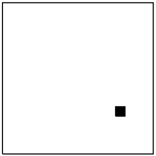

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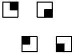

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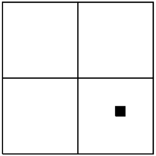

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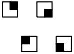
Flipping the orientation can leave hole anywhere. 

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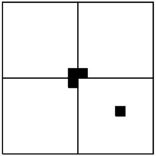

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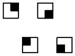

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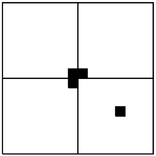

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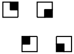

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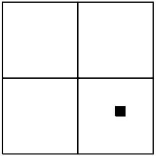

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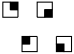

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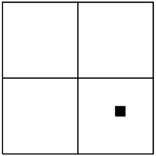

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(Contrapositive of Induction principle (assuming $P(0)$)

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
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E.g. Reduced form is "smallest" representation of rational number a/b.

Well ordering principle.

Thm: All natural numbers are interesting.

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Thus, there is no smallest uninteresting natural number.

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Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

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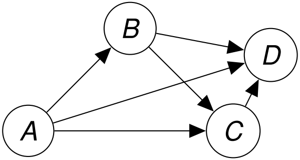

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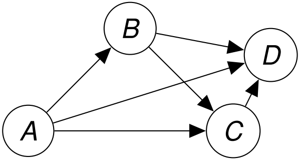

Theorem: Any tournament that has a cycle has a cycle of length 3.

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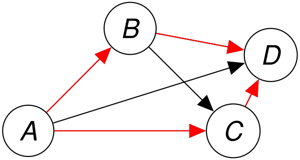

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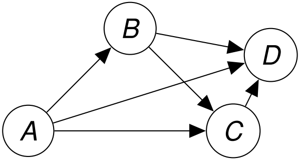

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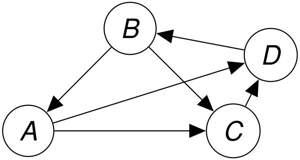

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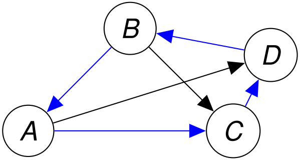

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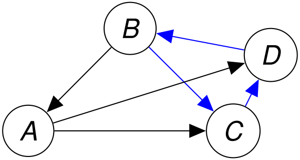

Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

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Assume the the **smallest cycle** is of length k.

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Case 1: Of length 3.

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Case 1: Of length 3. Done.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

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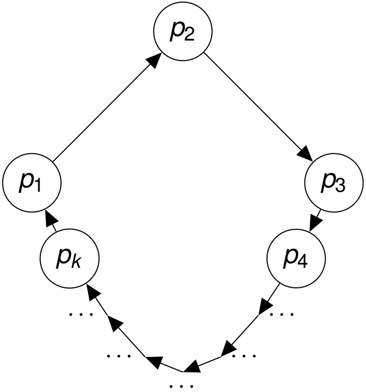

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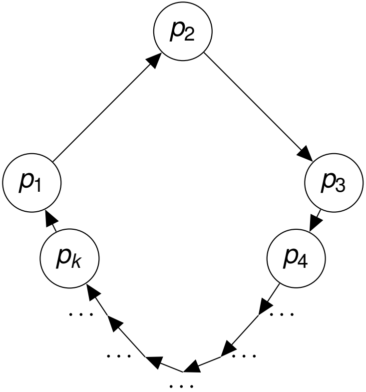

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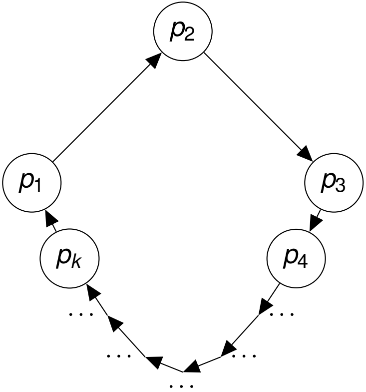

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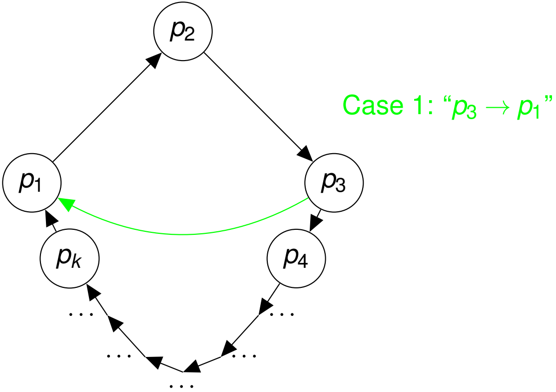

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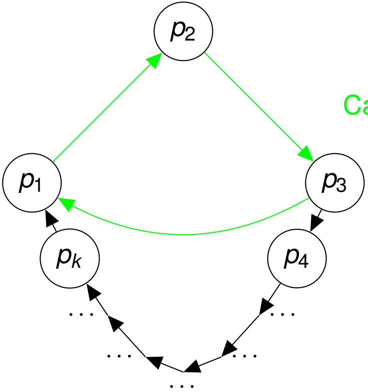

Case 1: "$p_3 \to p_1$" $\implies$ 3 cycle

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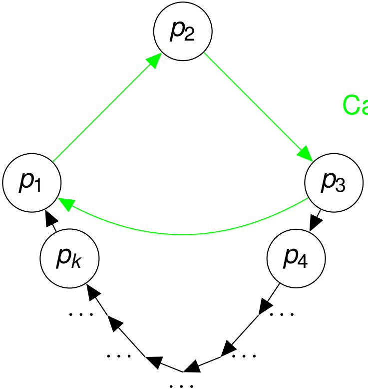

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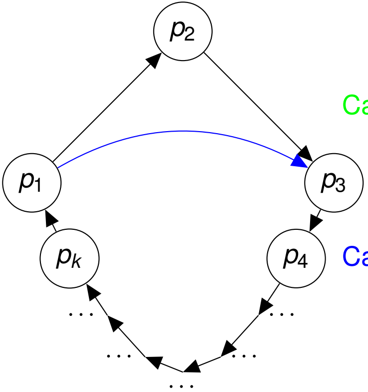

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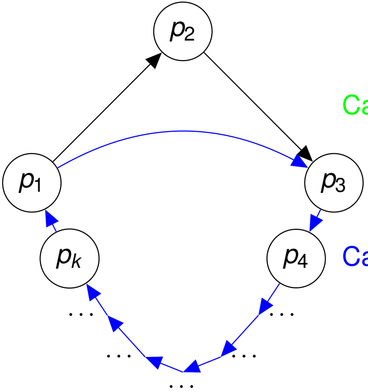

Case 1: "$p_3 \to p_1$" $\implies$ 3 cycle

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Case 2: "$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

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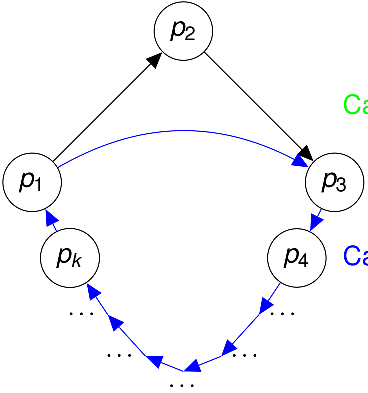

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Contradicts assumption of smallest k!

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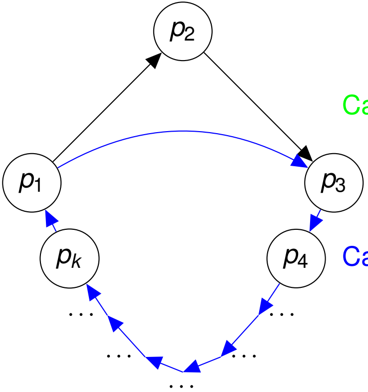

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Tournaments have long paths.

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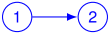

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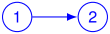

(Also for one, but two is more fun as base case!)

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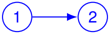

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Tournament on $n+1$ people,

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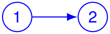

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Tournament on $n+1$ people,

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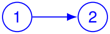

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Tournament on $n+1$ people,
Remove arbitrary person p $\rightarrow$ yield tournament on $n-1$ people.

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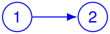

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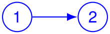

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Tournament on $n+1$ people,

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By induction hypothesis: There is a sequence $p_1, \ldots, p_n$

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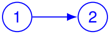

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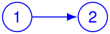

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Tournament on $n+1$ people,
Remove arbitrary person $p \to$ yield tournament on $n-1$ people.

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Thm: Every tournament has a Hamiltonian path.

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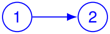

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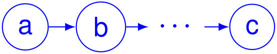

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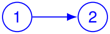

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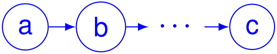

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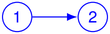

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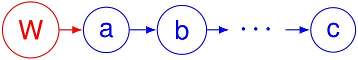

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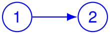

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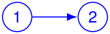

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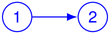

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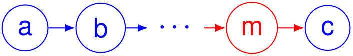

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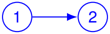

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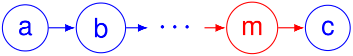

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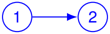

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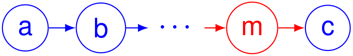

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Horses of the same color...

Theorem: All horses have the same color.

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As we will see, it is more subtle to catch errors in proofs of correct theorems!!

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Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Thm: If there are n villagers with green eyes they do ritual on day n.

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Wait a second! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

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On day 1, everyone knows everyone sees more than zero.

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Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

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On day 99, no one sees 98

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Related example:

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Related example:
Emperor's new clothes!

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

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Related example:
Emperor's new clothes!
  Maybe: No one knows other people see that he has no clothes.

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On day 1, everyone knows everyone sees more than zero.

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On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Related example:
Emperor's new clothes!
  Maybe: No one knows other people see that he has no clothes.
  Until kid points it out.

Summary: principle of induction.

Today: More induction.

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$

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$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

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Induction $\equiv$ Recursion.

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Example:
Coins of value 4 and 7, can be used to make any value higher than 11.
$P(\text{“}zero\text{''})$ here is $P(11)$, and prove for $\forall n \geq 11 \quad P(n) \implies P(n+1)$.