Today.

Principle of Induction.(continued.)

$P(0) \wedge (\forall n \in \mathbb{N}) P(n) \implies P(n+1)$

And we get...

$(\forall n \in \mathbb{N}) P(n)$.

...Yes for 0, and we can conclude Yes for 1...
and we can conclude Yes for 2.......

Last time.

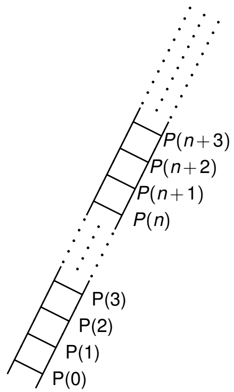
"But aren't you starting with the statement to prove the statement?"
In general, $\forall n \in N \quad P(n) \not\equiv P(n+1)$.
"$P(n)$" is "different" statement than "$P(n+1)$" for any fixed n.

$P(n) \equiv$ "$\sum_0^n i = \frac{(n)(n+1)}{2}$."

$P(2) \equiv$ "$0+1+2 = \frac{(2)(3)}{2}$." $\quad P(3) \equiv$ "$0+1+2+3 = \frac{(3)(4)}{2}$"

E.g. $\forall n \in N \quad P(n) \implies P(n+1)$.
Start with statement $P(n)$ and prove $P(n+1)$.
Is this a statement or a predicate?
A little subtle;
we fix the natural number n (bind n in $\forall n$.)
Assume $P(n)$ and then prove $P(n+1)$.
We did this before: $a|b$ and $a|c \implies a|b+c$.
How?
$b = ka$ and $c = k'a \implies (c+d) = (k+k')a$ or $a|(c+d)$.
Distributive property applies for any three integers.
Argument applies regardless of which numbers a, b, c are!
BTW: Can you do induction over the reals? No.
The notion of "next" is undefinable.

Climb an infinite ladder?



$P(0)$
$\forall k, P(k) \implies P(k+1)$
$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$
$(\forall n \in N) P(n)$

Your favorite example of forever..or the natural numbers...

Gauss and Induction

Child Gauss: $(\forall n \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$
$= \frac{k(k+1)}{2} + (k+1) = (k+1)(\frac{k}{2}+1) = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!
Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$
...
true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$
...

Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3-n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

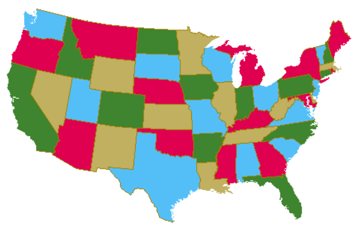
$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$
$= k^3 + 3k^2 + 2k$
$= (k^3 - k) + 3k^2 + 3k$ Subtract/add k
$= 3q + 3(k^2 + k)$ Induction Hyp. Factor.
$= 3(q + k^2 + k)$ (Un)Distributive + over $\times$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction. $\square$

Four Color Theorem.

Theorem: Any map can be colored so that those regions that share an edge have different colors.

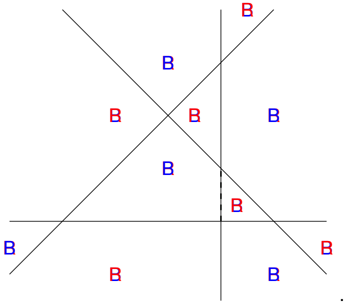


Check Out: "Four corners".
States connected at a point, can have same color.

Quick Test: Which states? Utah. Colorado. New Mexico. Arizona.

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.



Fact: Swapping red and blue gives another valid colors.

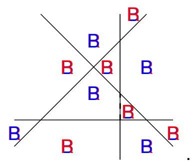
Swapping works.

What does works mean?

Gives another valid coloring?

What does valid mean?

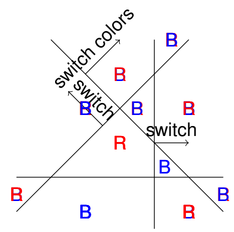
For each line segment or ray, any two regions of the plane that are separated by the ray or line segment are different colors.



Swapping a valid coloring works.

It was valid, so adjacent regions were different colors, changing both colors to each other, implies they are still different.

Two color theorem: proof illustration.



Base Case.
1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.
(Fixes conflicts along line, and makes no new ones.)

Algorithm gives $P(k) \implies P(k+1)$. □

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

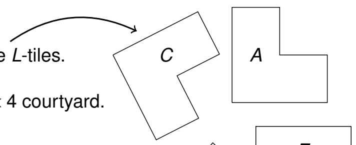
Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step:
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
   ????
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

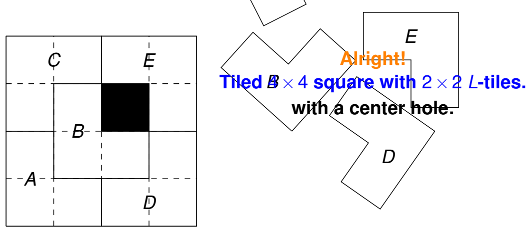
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



To Tile this 4×4 courtyard.



Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k=0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a+1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a+1) \\
&= 12a+3+1 \\
&= 3(4a+1)+1
\end{aligned}
$$

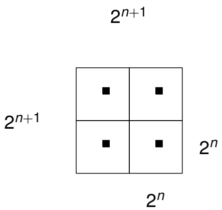
a integer $\implies (4a+1)$ is an integer. □

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.



What to do now???

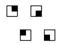
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

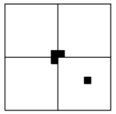
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. □

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors! □

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
I.e.,

$$(\neg\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Smallest may not be what you expect: the well ordering principal holds for rationals but with different ordering!!

E.g. Reduced form is "smallest" representation of rational number a/b.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...

Let n be the first uninteresting number.

But $n-1$ is interesting and n is uninteresting,

so n is the first uninteresting number.

But being first is interesting!

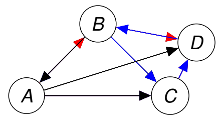
Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.



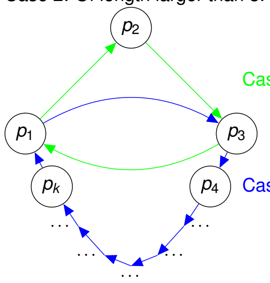
Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 1: "$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

Case 2: "$p_1 \to p_3$" $\implies$ $k-1$ length cycle!

Contradicts assumption of smallest k!

□

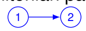
Tournaments have long paths.

Def: A **round robin tournament on** n **players**: all pairs p and q play, and either $p \to q$ (p beats q) or $q \to q$ (q beats q.)

Def: A **Hamiltonian path**: a sequence $p_1, \ldots, p_n$, $(\forall i, 0 \leq i < n)$ $p_i \to p_{i+1}$.

Thm: Every tournament has a Hamiltonian path.

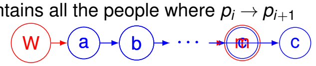
Base case: True for two vertices.



(Also for one, but two is more fun as base case!)

Tournament on $n+1$ people,
Remove arbitrary person $p \to$ yield tournament on $n-1$ people.

By induction hypothesis: There is a sequence $p_1, \ldots, p_n$ contains all the people where $p_i \to p_{i+1}$



If p is big winner, put at beginning. Big loser at end.
If neither, find first place i, where p beats p_i.
$p_1, \ldots, p_{i-1}, p, p_i, \ldots p_n$ is Hamiltonion path.

□

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1, 2, 2, 3, \ldots, k, k+1$
Last k have same color by $P(k)$. $1, 2, 2, 3, \ldots, k, k+1$
A horse in the middle in common! $1, 2, 2, 3, \ldots, k, k+1$
All k must have the same color. No horse in common! $1, 2, 3, \ldots, k, k+1$
How about $P(1) \implies P(2)$?
Fix base case.
There are two horses of the same color. ...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must do **bad ritual** that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don't talk about eye color!

Visitor: "I see someone has green eyes."

Result: On day 100, they all do the ritual.

Why?

They know induction.

Thm: If there are n villagers with green eyes they do ritual on day n.

Proof:
Base: $n = 1$. Person with green eyes does ritual on day 1.

Induction hypothesis:
If n people with green eyes, they do ritual on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't do the ritual.

So there must be $n+1$ people with green eyes.

One of them is me.

Sad.

□

Wait a second! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, no one sees 98 since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Related example:
Emperor's new clothes!
Maybe: No one knows other people see that he has no clothes.
Until kid points it out.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge (\forall n \in N)(P(0) \wedge P(1) \wedge \cdots P(n)) \implies P(n+1))))$
$\implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction. E.g., used in product of primes proof.

Induction $\equiv$ Recursion.

Summary: principle of induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Another Variation:

Different Starting Point:
$(P(1) \wedge ((\forall n \in N)((n \geq 1) \wedge P(n)) \implies P(n+1))))$
$\implies (\forall n \in N)((n \geq 1) \implies P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Example:
Coins of value 4 and 7, can be used to make any value higher than 11.
$P(\text{"zero"})$ here is $P(11)$, and prove for $\forall n \geq 11 \quad P(n) \implies P(n+1)$.