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Count with order.

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Add number of each and then subtract intersection of sets.

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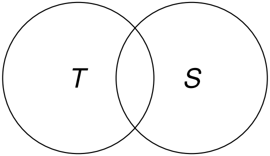

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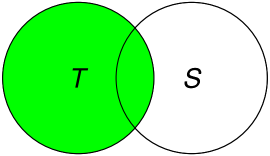

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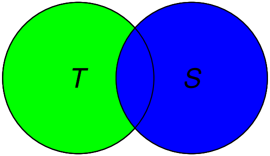

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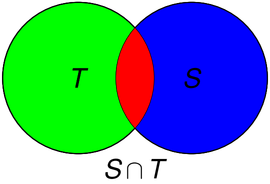

In T. $\implies$ $|T|$
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Elements in $S \cap T$ are counted twice.

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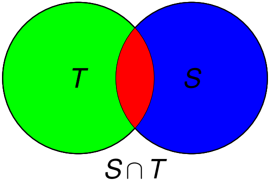

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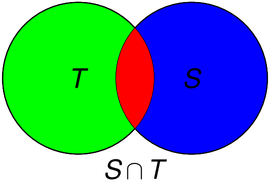

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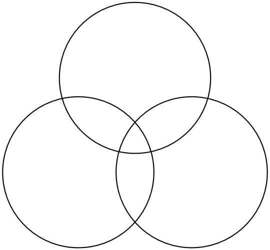

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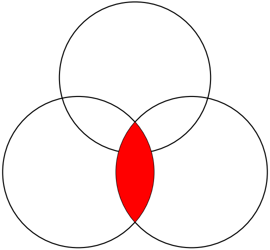

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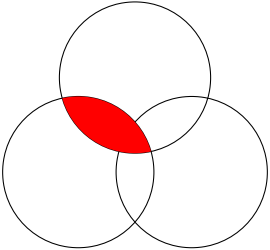

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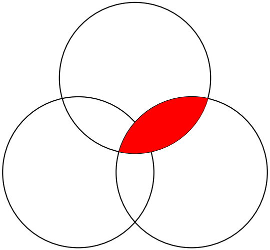

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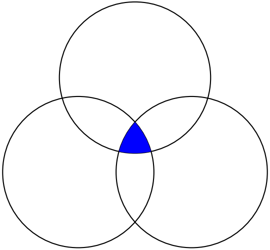

In general.

Sets $A_1, \ldots, A_n$

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$\implies 1 = \binom{k}{1} - \binom{k}{2} + \cdots + \binom{k}{k}(-1)^{k+1}$.

Derangements

Permutations of $1, \ldots, n$?

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$\binom{3}{1}(2!) - \binom{3}{2}(1) + \binom{3}{3}1.$

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$3! - 4 = 2.$

General Case: for fun!

For n items.

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Number of derangements is

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Term i is $\frac{n!}{(n-i)!i!(-1)^i} \times (n-i)!$

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$n! - \binom{n}{1}n-1! - \binom{n}{2}(n-1)! + \cdots + (-1)^{n-1}\binom{n}{n}$

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And get an analagous conclusion.

Sampling...

Sample k items out of n

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Without replacement:

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Without replacement:

Order matters:

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1$

Sampling...

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Without replacement:

Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders

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Second Rule: divide by number of orders – "$k!$"

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"n choose k"

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Order matters: $n \times n$

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How do we deal with this mess??

Splitting up some money....

How many ways can Bob and Alice split 5 dollars?

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For each of 5 dollars pick Bob or Alice(2^5), divide out order

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"Sorted" way to specify, first Alice's dollars, then Bob's.

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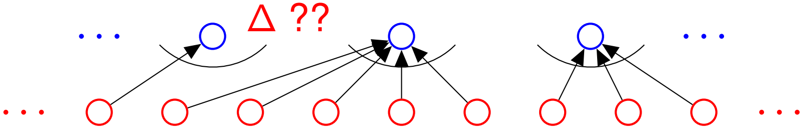

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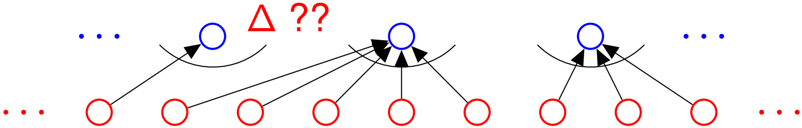

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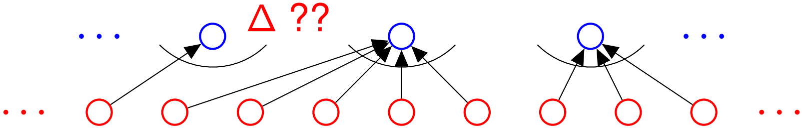
one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice:
5 ordered sets: (A, B, B, B, B) ; (B, A, B, B, B); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.

(B, B, B, B, B): 1: (B,B,B,B,B)
(A, B, B, B, B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A, A, B, B, B):

and so on.



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(B, B, B, B, B): 1: (B,B,B,B,B)
(A, B, B, B, B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A, A, B, B, B): $\binom{5}{2}$;(A,A,B,B,B),(A,B,A,B,B),(A,B,B,A,B),...

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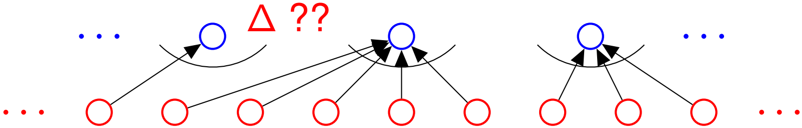

Second rule of counting is no good here!

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How many ways can Alice, Bob, and Eve split 5 dollars.

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Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

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Five dollars are five stars: $\star\star\star\star\star$.

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Alice: 2, Bob: 1, Eve: 2.

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Stars and Bars: $\star\star|\star|\star\star$.

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Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.

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Each split "is" a sequence of stars and bars.

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Five dollars are five stars: $\star\star\star\star\star$.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$.

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Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars.
Each sequence of stars and bars "is" a split.

Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \; \_ \; \_ \; \_ \; \_ \; \_ \; \_$

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

_ _ _ _ _ _ _

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_\ \_\ \_\ \_\ \_\ \_\ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.
Bars in second and seventh position.

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \_ \_ \_ \_ \_ \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.
Bars in second and seventh position.

$\binom{7}{2}$ ways to do so and

Stars and Bars.

How many different 5 star and 2 bar diagrams?

$| \star | \star \star \star \star$.

7 positions in which to place the 2 bars.

$\_ \; \_ \; \_ \; \_ \; \_ \; \_ \; \_$

Alice: 0; Bob 1; Eve: 4
$| \star | \star \star \star \star$.
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
$\star | \star \star \star \star |$.
Bars in second and seventh position.

$\binom{7}{2}$ ways to do so and
$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

Ways to add up n numbers to sum to k?

Stars and Bars.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

Stars and Bars.

Ways to add up *n* numbers to sum to *k*? or

" *k* from *n* with replacement where order doesn't matter."

Correspondence: how many times is item *i* in the sample.

Stars and Bars.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

Correspondence: how many times is item i in the sample.

Correspondence: k stars $n-1$ bars gives assignment to n numbers!

$$\star\star|\star|\cdots|\star\star.$$

Stars and Bars.

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$n+k-1$ positions from which to choose $n-1$ bar positions.

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$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Stars and Bars.

Ways to add up n numbers to sum to k? or

" k from n with replacement where order doesn't matter."

Correspondence: how many times is item i in the sample.

Correspondence: k stars $n-1$ bars gives assignment to n numbers!

$$\star\star|\star|\cdots|\star\star.$$

$n+k-1$ positions from which to choose $n-1$ bar positions.

$$\binom{n+k-1}{n-1}$$

Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$**.**

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter (sometimes) can divide...

Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

k Samples with replacement from n items: n^k.
Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter (sometimes) can divide...

Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
"n choose k"

Summary.

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pause Bijection!

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One-to-one rule: equal in number if one-to-one correspondence.
pause Bijection!

Sample k times from n objects with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

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5 balls into 10 bins

Balls in bins.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

Balls in bins.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order

Balls in bins.

"k Balls in n bins" $\equiv$ "k samples from n possibilities."

"indistinguishable balls" $\equiv$ "order doesn't matter"

"only one ball in each bin" $\equiv$ "without replacement"

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out

Combinatorial Proofs.

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How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: How many subsets of size k? $\binom{n}{k}$

How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.

Combinatorial Proofs.

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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How many subsets of size k?
Choose a subset of size $n-k$
and what's left out is a subset of size k.
Choosing a subset of size k is same
as choosing $n-k$ elements to not take.
$\implies \binom{n}{n-k}$ subsets of size k.

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□

Pascal's Triangle

Pascal's Triangle

0

1   1

Pascal's Triangle

0

1 1

1 2 1

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms)

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms:

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each factor of $(1+x)$.

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

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Foil (4 terms) on steroids:
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Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Foil (4 terms) on steroids:
2^n terms: choose 1 or x from each factor of $(1+x)$.

Simplify: collect all terms corresponding to x^k.
Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

$$
\begin{array}{ccc}
 & \binom{0}{0} & \\
\binom{1}{0} & & \binom{1}{1}
\end{array}
$$

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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$$\begin{array}{ccccc} & & \binom{0}{0} & & \\ & \binom{1}{0} & & \binom{1}{1} & \\ \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} \end{array}$$

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

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$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's Triangle

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

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Foil (4 terms) on steroids:
  2^n terms: choose 1 or x from each factor of $(1+x)$.

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  Coefficient of x^k is $\binom{n}{k}$: choose k factors where x is in product.

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

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Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?

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Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element,

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Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element,
need to choose $k-1$ more from remaining n elements.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element,
need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$

Combinatorial Proofs.

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Chose first element,
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$\implies \binom{n}{k-1}$

How many don't contain the first element ?

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

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How many contain the first element?
Chose first element,
need to choose $k-1$ more from remaining n elements.
$\implies \binom{n}{k-1}$

How many don't contain the first element ?
Need to choose k elements from remaining n elts.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$? $\binom{n+1}{k}$.

How many size k subsets of $n+1$?
How many contain the first element?
Chose first element,
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Need to choose k elements from remaining n elts.
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Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

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Plugin $x = 1, y = 1$ or $(1+1)^n$ into Theorem A.

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