Today.

Finish up undecidability.
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Counting.
Programs and Diagonalization.

Write me a program checker!

Check that the compiler works!

How about.. Check that the compiler terminates on a certain input.

\[ \text{HALT} (P, I) \]

\( P \)- program
\( I \)- input.

Determines if \( P(I) \) (run on \( I \)) halts or loops forever.

Notice: Need a computer...with the notion of a stored program!!!! (not an adding machine! not a person and an adding machine.)

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Text string can be an input to a program.
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How long do you wait?

Something about infinity here, maybe?
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$HALT(P, I)$
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\[ \text{HALT}(P, I) \]

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(A) He is confused.
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Halt and Turing.

Proof:

Assume there is a program $\text{HALT}(\cdot, \cdot)$. Turing($P$)

1. If $\text{HALT}(P, P) = \text{halts}$, then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program $\text{HALT}$.

There is text that is the program $\text{HALT}$.

There is text that is the program $\text{Turing}$.

Can run $\text{Turing}$ on $\text{Turing}$!

Does $\text{Turing}(\text{Turing})$ halt?

$\text{Turing}(\text{Turing})$ halts $\Rightarrow \text{HALTS}(\text{Turing}, \text{Turing}) = \text{halts}$ $\Rightarrow \text{Turing}(\text{Turing})$ loops forever.

$\text{Turing}(\text{Turing})$ loops forever $\Rightarrow \text{HALTS}(\text{Turing}, \text{Turing}) \neq \text{halts}$ $\Rightarrow \text{Turing}(\text{Turing})$ halts.

Contradiction.

Program $\text{HALT}$ does not exist!

Questions?
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot,\cdot)$. 
Halt and Turing.

Proof: Assume there is a program $HALT(\cdot, \cdot)$.

Turing($P$)
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**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing(P)
1. If $HALT(P,P) =$“halts”, then go into an infinite loop.
Halt and Turing.

**Proof:** Assume there is a program $HALT(·, ·)$.

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Contradiction. Program $HALT$ does not exist!
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Contradiction. Program HALT does not exist!

Questions?
Another view of proof: diagonalization.

Any program is a fixed length string.
Another view of proof: diagonalization.

Any program is a fixed length string. Fixed length strings are enumerable.
Another view of proof: diagonalization.

Any program is a fixed length string.  
Fixed length strings are enumerable.  
Program halts or not any input, which is a string.
Another view of proof: diagonalization.

Any program is a fixed length string. 
Fixed length strings are enumerable. 
Program halts or not any input, which is a string.

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<td>( P_1 )</td>
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<td>H</td>
<td>L</td>
<td>...</td>
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<tr>
<td>( P_2 )</td>
<td>L</td>
<td>L</td>
<td>H</td>
<td>...</td>
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<td>( P_3 )</td>
<td>L</td>
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Another view of proof: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
Program halts or not any input, which is a string.

|     |  $P_1$ |  $P_2$ |  $P_3$ |   ...
|-----|-------|-------|-------|-------|
| $P_1$ |  H    |  H    |  L    |   ...
| $P_2$ |  L    |  L    |  H    |   ...
| $P_3$ |  L    |  H    |  H    |   ...
|   .. |   ..  |   ..  |   ..  |   ..  |

Halt - diagonal.
Another view of proof: diagonalization.

Any program is a fixed length string.  
Fixed length strings are enumerable.  
Program halts or not any input, which is a string.

\[
\begin{array}{c|cccc}
 & P_1 & P_2 & P_3 & \ldots \\
P_1 & H & H & L & \ldots \\
P_2 & L & L & H & \ldots \\
P_3 & L & H & H & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\hline
\end{array}
\]

Halt - diagonal.  
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Another view of proof: diagonalization.

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<td>H</td>
<td>H</td>
<td>L</td>
<td>...</td>
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<tr>
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<td>L</td>
<td>L</td>
<td>H</td>
<td>...</td>
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<tr>
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<td>H</td>
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Halt - diagonal. Turing - is not Halt. and is different from every $P_i$ on the diagonal. Turing is not on list.
Another view of proof: diagonalization.

Any program is a fixed length string.
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Program halts or not any input, which is a string.

\[
\begin{array}{c|cccc}
 & P_1 & P_2 & P_3 & \ldots \\
 P_1 & H & H & L & \ldots \\
P_2 & L & L & H & \ldots \\
P_3 & L & H & H & \ldots \\
: & : & : & : & : \\
\end{array}
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Halt does not exist!
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Halt does not exist!
Discussion of Proof.

Undecidability:
Discussion of Proof.

Undecidability:
\[ \text{HALT}(P) - \text{does not exist.} \]
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Undecidability:
   \( \text{HALT}(P) \) - does not exist.

Why not?
Discussion of Proof.

Undecidability:
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Why not?
Discussion of Proof.

Undecidability:
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Why not?

Programs are text.
Undecidability:
  HALT(P) - does not exist.

Why not?

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Why not?

Programs are text.

List programs,
Discussion of Proof.

Undecidability:
  \( \text{HALT}(P) \) - does not exist.

Why not?

Programs are text.

List programs, Turing not in list of programs!
Discussion of Proof.

Undecidability:
   \text{HALT}(P) - does not exist.

Why not?

Programs are text.

List programs, Turing not in list of programs!

Argue directly by saying \text{Turing}(\text{Turing}) neither halts nor runs forever.
Discussion of Proof.

Undecidability:
   \( \text{HALT}(P) \) - does not exist.

Why not?

Programs are text.

List programs, Turing not in list of programs!

Argue directly by saying Turing(Turing) neither halts nor runs forever.

Really Same: Says there is no text which can be Turing.
Fairly quick. Right?
Notes.

Fairly quick. Right?
No computers in early 20th century.
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Turing's proof came up with the notion of computer.
Notes.

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Also, the incompleteness theorems also designed a computer using arithmetic.
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CS 61C + one slide
Notes.

Fairly quick. Right?
No computers in early 20th century.
Turing's proof came up with the notion of computer.
Also, the incompleteness theorems also designed a computer using arithmetic.
CS 61C + one slide $\implies$ undecidability of halting.
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”?
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How?
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How? What is $P$?
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Does a program, $P$, print “Hello World”? How? What is $P$? Text!!!!!
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Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How? What is $P$? Text!!!!!!

Solve $\text{HALT}(P,I)$:
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How? What is $P$? Text!!!!!!

Solve $\text{HALT}(P,I)$:
  Make $P'$ as follows:
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How? What is $P$? Text!!!!!!

Solve $\text{HALT}(P,I)$:
   Make $P'$ as follows:
      Remove all print statements.
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”? How? What is $P$? Text!!!!!!

Solve $\text{HALT}(P,I)$:
Make $P'$ as follows:
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Undecidable questions about Programs.

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  Make $P'$ as follows:
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Call $\text{PrintsHelloWorld}(P',I)$
Undecidable questions about Programs.

Does a program, \( P \), print “Hello World”? How? What is \( P \)? Text!!!!!!

Solve \( \text{HALT}(P,I) \):

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- Remove all print statements.
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\( P \) halts if and only if \( P' \) prints hello world.
Undecidable questions about Programs.

Does a program, $P$, print “Hello World”?
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Many things one can ask about programs that are undecidable.
Undecidable questions about Programs.

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Many things one can ask about programs that are undecidable.
Because programs are text.
Other example undecidable problems.

Can a set of notched tiles tile the infinite plane?
Other example undecidable problems.

Can a set of notched tiles tile the infinite plane?
Proof: simulate a computer. Halts if finite.
Other example undecidable problems.

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Can a set of notched tiles tile the infinite plane? Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution?

Example: 

\[ x^n + y^n = 1? \]

Problem is undecidable.

Be careful!

Is there an integer solution to 

\[ x^n + y^n = 1? \]

(Diophantine equation.)

The answer is yes or no.

Undecidability for Diophantine set of equations 

\[ \Rightarrow \]

no program can take any set of integer equations and always correctly output whether it has an integer solution.
Other example undecidable problems.

Can a set of notched tiles tile the infinite plane? Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution? Example: “$x^n + y^n = 1$?”
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Undecidability for Diophantine set of equations
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Undecidability for Diophantine set of equations
$\implies$ no program can take any set of integer equations and
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Undecidability for Diophantine set of equations

$\implies$ no program can take any set of integer equations and always correctly output whether it has an integer solution.
Comments: undecidability.

Computer Programs are an interesting thing.
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Like Math.
Formal Systems.
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Computer Programs cannot completely “understand” computer programs.
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Proof Idea:
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Proof Idea: Diagonalization.
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Proof Idea: Diagonalization.  
Program: Turing (or DIAGONAL) takes $P$.  

Computer Programs are an interesting thing. Like Math. Formal Systems.

Computer Programs cannot completely “understand” computer programs.

Example: no computer program can tell if any other computer program HALTS.

Proof Idea: Diagonalization.
Program: Turing (or DIAGONAL) takes $P$. Assume there is HALT.
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Proof Idea: Diagonalization.
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   DIAGONAL flips answer: "Loops if $P$ halts, halts if $P$ loops."
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What does Turing do on turing? Doesn’t loop or HALT.
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Computation as a lens

Computation is a lens for other action in the world.
Computation as a lens

Computation is a lens for other action in the world.

E.g. Turing’s work on linear systems (condition number), chemical networks (embryo.)
Computation as a lens

Computation is a lens for other action in the world.

E.g. Turing’s work on linear systems (condition number), chemical networks (embryo.)

Today: Quantum computing, evolution models, models of the brain, complexity of Nash equilibria, ...
Your future (in this course).

What’s to come?
Your future (in this course).

What’s to come? Probability.
What’s to come? Probability.

A bag contains:
Your future (in this course).

What’s to come? Probability.

A bag contains:

![Red, blue, and yellow circles](image-url)
Your future (in this course).

What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?
Your future (in this course).

What’s to come? Probability.

A bag contains:

- Red
- Blue
- Yellow
- Blue
- Red
- Red
- Red
- Blue

What is the chance that a ball taken from the bag is blue?
Count blue.
What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total.
Your future (in this course).

What’s to come? Probability.

A bag contains:

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Your future (in this course).

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Today:
Your future (in this course).

What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?


Today: Counting!
Your future (in this course).

What’s to come? Probability.

A bag contains:

![Bag contents: 3 red, 5 blue, 2 yellow balls](image)

What is the chance that a ball taken from the bag is blue?


Today: Counting!

Later: Probability.
Your future (in this course).

What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?


Today: Counting!

Later: Probability. Professor Ayazifar.
What’s to come? Probability.

A bag contains:

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Your future (in this course).

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A bag contains:

What is the chance that a ball taken from the bag is blue?


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Babak
Your future (in this course).

What’s to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?


Today: Counting!


Babak ≡ “Bob” Back.
Outline: basics

1. Counting.
2. Tree
3. Rules of Counting
4. Sample with/without replacement where order does/doesn’t matter.
Probability is soon..but first let’s count.
Count?

How many outcomes possible for $k$ coin tosses?
How many poker hands?
How many handshakes for $n$ people?
How many diagonals in a convex polygon?
How many 10 digit numbers?
How many 10 digit numbers without repetition?
Using a tree..

How many 3-bit strings?
Using a tree..

How many 3-bit strings?
How many different sequences of three bits from \{0, 1\}?
Using a tree.

How many 3-bit strings?
How many different sequences of three bits from \{0, 1\}? How would you make one sequence?
Using a tree..

How many 3-bit strings?
How many different sequences of three bits from \{0, 1\}?
How would you make one sequence?
How many different ways to do that making?
How many 3-bit strings?
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How many 3-bit strings?
How many different sequences of three bits from \( \{0, 1\} \)?
How would you make one sequence?
How many different ways to do that making?

\[
\begin{align*}
&0 \\
&1 \\
\end{align*}
\]
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Using a tree..

8 leaves which is \( 2 \times 2 \times 2 \).

One leaf for each string.

8 3-bit strings!
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000 001 010 011 100 101 110 111
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First Rule of Counting: Product Rule

Objects made by choosing from $n_1$, then $n_2$, ..., then $n_k$ the number of objects is $n_1 \times n_2 \cdots \times n_k$. 
First Rule of Counting: Product Rule

Objects made by choosing from $n_1$, then $n_2$, ..., then $n_k$ the number of objects is $n_1 \times n_2 \cdots \times n_k$. 

\[
\begin{array}{c}
\text{Diagram}
\end{array}
\]
First Rule of Counting: Product Rule

Objects made by choosing from \( n_1 \), then \( n_2, \ldots, \) then \( n_k \) the number of objects is \( n_1 \times n_2 \cdots \times n_k \).
First Rule of Counting: Product Rule

Objects made by choosing from $n_1$, then $n_2$, ..., then $n_k$ the number of objects is $n_1 \times n_2 \cdots \times n_k$. 

![Diagram of tree structure with branches leading to $n_1$, $n_2$, and $n_3$.]
First Rule of Counting: Product Rule

Objects made by choosing from \( n_1 \), then \( n_2 \), \ldots, then \( n_k \) the number of objects is \( n_1 \times n_2 \cdots \times n_k \).
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In picture, $2 \times 2 \times 3 = 12!$
First Rule of Counting: Product Rule

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the number of objects is $n_1 \times n_2 \cdots \times n_k$.

In picture, $2 \times 2 \times 3 = 12!$
Using the first rule..

How many outcomes possible for $k$ coin tosses?

$2 \times 2 \times \cdots \times 2 = 2^k$

How many 10 digit numbers?

$10 \times 10 \times \cdots \times 10 = 10^k$

How many $n$ digit base $m$ numbers?

$m \times m \times \cdots \times m = m^n$

(Is 09, a two digit number?)

If no. Then $(m - 1)^{m^n - 1}$.
Using the first rule..

How many outcomes possible for $k$ coin tosses?
2 ways for first choice,
Using the first rule..

How many outcomes possible for $k$ coin tosses?
2 ways for first choice, 2 ways for second choice, ...
Using the first rule.

How many outcomes possible for $k$ coin tosses?
2 ways for first choice, 2 ways for second choice, ...
$2^k$
Using the first rule..

How many outcomes possible for $k$ coin tosses?
2 ways for first choice, 2 ways for second choice, ... $2 \times 2$
Using the first rule..

How many outcomes possible for \( k \) coin tosses?
2 ways for first choice, 2 ways for second choice, ...
\( 2 \times 2 \cdots \)
Using the first rule..

How many outcomes possible for $k$ coin tosses?
2 ways for first choice, 2 ways for second choice, ...
$2 \times 2 \cdots \times 2$
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10 ways for first choice,
Using the first rule..

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How many $n$ digit base $m$ numbers?
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Using the first rule..

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\[ m^n \]
(Is 09, a two digit number?)
If no. Then \((m - 1)m^{n-1}\).
Functions, polynomials.

How many functions $f$ mapping $S$ to $T$?
Functions, polynomials.

How many functions $f$ mapping $S$ to $T$?

$|T|$ ways to choose for $f(s_1)$,
Functions, polynomials.

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$....|T|^{|S|}$
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How many functions \( f \) mapping \( S \) to \( T \)?

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\( \ldots |T|^{|S|} \)

How many polynomials of degree \( d \) modulo \( p \)?
Functions, polynomials.

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....$|T|^{|S|}$

How many polynomials of degree $d$ modulo $p$?
$p$ ways to choose for first coefficient,
Functions, polynomials.

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$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ... ....$|T|^{|S|}$

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$p$ ways to choose for first coefficient, $p$ ways for second, ...
Functions, polynomials.

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....\(| T |^{|S|}

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\( p \) ways to choose for first coefficient, \( p \) ways for second, ...

...\( p^{d+1} \)
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$p$ values for first point,
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$p$ values for first point, $p$ values for second, ...
...$p^{d+1}$

Questions?
Permutations.

How many 10 digit numbers without repeating a digit?

10 ways for first, 9 ways for second, 8 ways for third, ...

\[10 \times 9 \times 8 \cdots 1 = 10!\.

How many different samples of size \(k\) from \(n\) numbers without replacement?

\(n\) ways for first choice, \(n-1\) ways for second, \(n-2\) choices for third, ...

\[n \times (n-1) \times (n-2) \cdots (n-k+1) = \frac{n!}{(n-k)!}\.

How many orderings of \(n\) objects are there?

Permutations of \(n\) objects.

\(n\) ways for first, \(n-1\) ways for second, \(n-2\) ways for third, ...

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\(^1\)By definition: 0! = 1.
Permutations.

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\[n \text{ ways for first choice, } n - 1 \text{ ways for second, } n - 2 \text{ for third, \ldots } \]

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\(^1\)By definition: \(0! = 1\).
Permutations.

How many 10 digit numbers \textbf{without repeating a digit}? 
10 ways for first, 9 ways for second, 8 ways for third, ... 
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How many different samples of size $k$ from $n$ numbers \textbf{without replacement}.

\hspace{2.3cm} 10 \times 9 \times 8 \times \cdots \times 1 = 10!.\footnote{By definition: $0! = 1$.}
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\(n\) ways for first, \(n - 1\) ways for second, \(n - 2\) ways for third, ...

... \(n \cdot (n - 1) \cdot (n - 2) \cdot 1 = n!\).

\(^1\)By definition: 0! = 1.
One-to-One Functions.

How many one-to-one functions from $|S|$ to $|S|$. $|S|$ choices for $f(s_1)$, $|S| - 1$ choices for $f(s_2)$, ... So total number is $|S| \times (|S| - 1) \cdots 1 = |S|!$.

A one-to-one function is a permutation!
One-to-One Functions.

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A one-to-one function is a permutation!
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$|S|$ choices for $f(s_1)$,
One-to-One Functions.

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So total number is $|S| \times |S| - 1 \cdots 1 = |S|!$

A one-to-one function is a permutation!
Counting sets...when order doesn’t matter.

How many poker hands?

\[
\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}
\]

Can write as...

\[
\frac{52!}{5! \times 47!}
\]

Generic: ways to choose 5 out of 52 possibilities.

\(^2\)When each unordered object corresponds equal numbers of ordered objects.
Counting sets..when order doesn’t matter.

How many poker hands?

52 × 51 × 50 × 49 × 48

A, K, Q, 10, J of spades and 10, J, Q, K, A of spades the same?

Second Rule of Counting:
If order doesn’t matter count ordered objects and then divide by number of orderings.

Number of orderings for a poker hand: “5!”

(The “!” means factorial, not Exclamation.)

Can write as...

52!

5!

47!

Generic: ways to choose 5 out of 52 possibilities.

When each unordered object corresponds equal numbers of ordered objects.

\[2\text{When each unordered object corresponds equal numbers of ordered objects.}\]
Counting sets..when order doesn’t matter.

How many poker hands?

\[ 52 \times 51 \times 50 \times 49 \times 48 \ ??? \]

Second Rule of Counting:
If order doesn’t matter count ordered objects and then divide by number of orderings.

Number of orderings for a poker hand: "5!"

(The "!" means factorial, not Exclamation.)

\[ \frac{52!}{5! \times 47!} \]

Can write as...

\[ \binom{52}{5} \]

Generic: ways to choose 5 out of 52 possibilities.

\(^2\)When each unordered object corresponds equal numbers of ordered objects.
Counting sets...when order doesn't matter.

How many poker hands?

\[52 \times 51 \times 50 \times 49 \times 48 \ ???\]

Are \(A, K, Q, 10, J\) of spades
and \(10, J, Q, K, A\) of spades the same?

\(^2\text{When each unordered object corresponds equal numbers of ordered objects.}\)
Counting sets..when order doesn’t matter.

How many poker hands?

\[ 52 \times 51 \times 50 \times 49 \times 48 \]

Are \( A, K, Q, 10, J \) of spades
and \( 10, J, Q, K, A \) of spades the same?

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\(^2\)When each unordered object corresponds equal numbers of ordered objects.
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**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.\(^2\)

Number of orderings for a poker hand: “5!”
(The “!” means factorial, not Exclamation.)

\[\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 47 \times 46 \times 45 \times 44} \]

\(^2\)When each unordered object corresponds equal numbers of ordered objects.
Counting sets..when order doesn’t matter.

How many poker hands?

\[52 \times 51 \times 50 \times 49 \times 48 \ ???\]

Are \(A, K, Q, 10, J\) of spades and \(10, J, Q, K, A\) of spades the same?

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\[
\frac{52 \times 51 \times 50 \times 49 \times 48}{5!}
\]

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Generic: ways to choose 5 out of 52 possibilities.

---

\(^2\)When each unordered object corresponds equal numbers of ordered objects.
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

How many red nodes (ordered objects)? 9.
How many red nodes mapped to one blue node? 3.
How many blue nodes (unordered objects)? 9.

How many poker deals? $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$.
How many poker deals per hand? Map each deal to ordered deal: $5!$.

How many poker hands? $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 / 5!$.
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

How many red nodes (ordered objects)?

How many red nodes mapped to one blue node?

How many blue nodes (unordered objects)?

How many poker deals?

How many poker deals per hand?

Map each deal to ordered deal:

How many poker hands?
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

How many red nodes (ordered objects)? 9.

How many red nodes mapped to one blue node? 3.

How many blue nodes (unordered objects)? 9

\[ \frac{9}{3} = 3 \]

How many poker deals?

\[ 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \]

How many poker deals per hand?

Map each deal to ordered deal:

\[ 5! \]

How many poker hands?

\[ 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \div 5! \]

Questions?
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

How many red nodes (ordered objects)? 9.
How many red nodes mapped to one blue node?
Ordered to unordered.

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Ordered to unordered.

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How many blue nodes (unordered objects)? \( \frac{9}{3} \).
Ordered to unordered.

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How many poker deals?

\[52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \] How many poker hands?

\[5\]
Second Rule of Counting: If order doesn’t matter count ordered objects and then divide by number of orderings.

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How many red nodes (ordered objects)? 9.

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How many blue nodes (unordered objects)? \( \frac{9}{3} = 3 \).

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How many poker deals per hand?
Ordered to unordered.

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Map each deal to ordered deal: \( 5! \).

How many poker hands? \( \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} \)
Ordered to unordered.

**Second Rule of Counting:** If order doesn’t matter count ordered objects and then divide by number of orderings.

How many red nodes (ordered objects)? 9.

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How many poker deals per hand? Map each deal to ordered deal: \( 5! \).

How many poker hands? \( \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} \).

Questions?
order doesn’t matter.
..order doesn’t matter.

Choose 2 out of \( n \)?
..order doesn’t matter.

Choose 2 out of $n$?

$$n \times (n-1)$$
..order doesn’t matter.

Choose 2 out of $n$?

$$\frac{n \times (n-1)}{2}$$
..order doesn’t matter.

Choose 2 out of $n$?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$
..order doesn’t matter.

Choose 2 out of \( n \)?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of \( n \)?

Notation: \( \binom{n}{k} \) and pronounced "\( n \) choose \( k \)."
..order doesn’t matter.

Choose 2 out of \( n \)?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of \( n \)?

\[
\frac{n \times (n-1) \times (n-2)}{(n-1) \times (n-2)}
\]
order doesn’t matter.

Choose 2 out of $n$?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of $n$?

\[
\frac{n \times (n-1) \times (n-2)}{3!}
\]
..order doesn’t matter.

Choose 2 out of $n$?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of $n$?

\[
\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}
\]

Notation: \((n \choose k)\) and pronounced "$n$ choose $k$." Familiar? Questions?
..order doesn’t matter.

Choose 2 out of $n$?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of $n$?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose $k$ out of $n$?

$$\frac{n!}{(n-k)!}$$

Notation: $\binom{n}{k}$ and pronounced "$n$ choose $k$.

Familiar? Questions?
order doesn’t matter.

Choose 2 out of $n$?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of $n$?

\[
\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}
\]

Choose $k$ **out of** $n$?

\[
\frac{n!}{(n-k)!}
\]
..order doesn’t matter.

Choose 2 out of $n$?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of $n$?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose $k$ out of $n$?

$$\frac{n!}{(n-k)! \times k!}$$
..order doesn’t matter.

Choose 2 out of \(n\)?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of \(n\)?

\[
\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}
\]

Choose \(k\) out of \(n\)?

\[
\frac{n!}{(n-k)! \times k!}
\]

Notation: \(\binom{n}{k}\) and pronounced “\(n\) choose \(k\).”
..order doesn’t matter.

Choose 2 out of \( n \)?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of \( n \)?

\[
\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}
\]

Choose \( k \) out of \( n \)?

\[
\frac{n!}{(n-k)! \times k!}
\]

Notation: \( \binom{n}{k} \) and pronounced “\( n \) choose \( k \).”

Familiar?
..order doesn’t matter.

Choose 2 out of \( n \)?

\[
\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}
\]

Choose 3 out of \( n \)?

\[
\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}
\]

Choose \( k \) **out of** \( n \)?

\[
\frac{n!}{(n-k)! \times k!}
\]

**Notation:** \( ^nC_k \) and pronounced “\( n \) choose \( k \)”

Familiar? Questions?
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

\[ \binom{52}{3} = \frac{52!}{49!3!}. \]

First rule again.

Total: \( n! \left( \begin{array}{c} n-k \end{array} \right) \frac{k!}{2!} \) (By first rule!)

Second rule.
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: 52
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \)
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 \)
Example: Visualize the proof..

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. 
Example: Visualize the proof..

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?
Example: Visualize the proof.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta ? \)
   Hand: \( Q, K, A \).
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta ? \)
   Hand: \( Q, K, A \).
   Deals: \( Q, K, A \):
Example: Visualize the proof..

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: $\Delta$?
   Hand: $Q, K, A$.
   Deals: $Q, K, A : Q, A, K$ :
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?
Hand: \( Q, K, A \).
Example: Visualize the proof..

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

**Second rule:** when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

**Hand:** \( Q, K, A \).


\( \Delta = 3 \times 2 \times 1 \)
Example: Visualize the proof..

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.

**Second rule:** when order doesn’t matter divide...

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

- **Hand:** Q, K, A.

\( \Delta = 3 \times 2 \times 1 \) First rule again.
Example: Visualize the proof.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.

**Second rule:** when order doesn’t matter divide...

\[
\begin{align*}
\cdots & \quad \Delta \\
\cdots & \quad \quad \quad \quad \quad \quad \quad \\
\cdots & \quad \quad \quad \quad \quad \quad \quad \\
\cdots & \quad \quad \quad \quad \quad \quad \quad \\
\cdots & \quad \quad \quad \quad \quad \quad \quad \\
\end{align*}
\]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

**Hand:** \( Q, K, A \).


\( \Delta = 3 \times 2 \times 1 \) First rule again.

Total:
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

\[
\Delta = 3 \times 2 \times 1 \text{ First rule again.}
\]

Total: \( \frac{52!}{49!3!} \)

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?

Hand: Q, K, A.

\( \Delta = 3 \times 2 \times 1 \) First rule again.
Example: Visualize the proof.

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Second rule: when order doesn’t matter divide...

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta ? \)

Hand: Q, K, A.

\( \Delta = 3 \times 2 \times 1 \) First rule again.
Total: \( \frac{52!}{49!3!} \) Second Rule!
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

\[
\Delta = 3 \times 2 \times 1 \quad \text{First rule again.}
\]

\[
\text{Total: } \frac{52!}{49!3!} \quad \text{Second Rule!}
\]

Choose \( k \) out of \( n \).
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?

Hand: \( Q, K, A \).

\( \Delta = 3 \times 2 \times 1 \) First rule again.
Total: \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).
Ordered set: \( \frac{n!}{(n-k)!} \)
Example: Visualize the proof..

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: $\Delta$?
Hand: Q, K, A.
$\Delta = 3 \times 2 \times 1$ First rule again.
Total: $\frac{52!}{49!3!}$ Second Rule!

Choose $k$ out of $n$.
Ordered set: $\frac{n!}{(n-k)!}$ Orderings of one hand?
Example: Visualize the proof.

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta ? \)
Hand: Q, K, A.
\( \Delta = 3 \times 2 \times 1 \) First rule again.
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Ordered set: \( \frac{n!}{(n-k)!} \) Orderings of one hand? \( k! \)
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**Second rule:** when order doesn’t matter divide...

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.

Poker hands: $\Delta$?
- **Hand:** $Q, K, A$.

$\Delta = 3 \times 2 \times 1$ First rule again.

Total: $\frac{52!}{49!3!}$ Second Rule!

Choose $k$ out of $n$.
- Ordered set: $\frac{n!}{(n-k)!}$ Orderings of one hand? $k!$ (By first rule!)
Example: Visualize the proof..

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

**Second rule:** when order doesn’t matter divide...

\[ \Delta \]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.

Poker hands: \( \Delta \)?

**Hand:** Q, K, A.


\[ \Delta = 3 \times 2 \times 1 \] First rule again.

Total: \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).

**Ordered set:** \( \frac{n!}{(n-k)!} \) Orderings of one hand? \( k! \) (By first rule!)

\[ \implies \text{Total: } \frac{n!}{(n-k)!k!} \]
Example: Visualize the proof..

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**

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Poker hands: \( \Delta \)?

**Hand:** \( Q, K, A \).


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\[ \implies \text{Total: } \frac{n!}{(n-k)!k!} \text{ Second rule.} \]
Example: Visualize the proof..

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

\[
\begin{align*}
\Delta & = 3 \times 2 \times 1 \text{ First rule again.} \\
\text{Total: } & \frac{52!}{49!3!} \text{ Second Rule!}
\end{align*}
\]

3 card Poker deals: \( 52 \times 51 \times 50 = \frac{52!}{49!} \). First rule.
Poker hands: \( \Delta \)?

Hand: \( Q, K, A \).
\( \Delta = 3 \times 2 \times 1 \) First rule again.
Total: \( \frac{52!}{49!3!} \) Second Rule!

Choose \( k \) out of \( n \).
Ordered set: \( \frac{n!}{(n-k)!} \) Orderings of one hand? \( k! \) (By first rule!)
\[ \Longrightarrow \text{Total: } \frac{n!}{(n-k)!k!} \text{ Second rule.} \]
Example: Anagram

First rule: $n_1 \times n_2 \cdots \times n_3$. Product Rule.
Second rule: when order doesn’t matter divide...
Example: Anagram

First rule: \( n_1 \times n_2 \cdots \times n_3 \). **Product Rule.**
Second rule: when order doesn’t matter divide...

Orderings of ANAGRAM?

\[ \Delta = 3 \times 2 \times 1 = 3! \]
\[ \Rightarrow 7! / 3! \]
Example: Anagram

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

Orderings of ANAGRAM?
Ordered Set: 7!
Example: Anagram

First rule: \( n_1 \times n_2 \cdots \times n_3 \). Product Rule.
Second rule: when order doesn’t matter divide...

Orderings of ANAGRAM?
Ordered Set: 7! First rule.
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ANAGRAM
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\( A_1 N A_2 G R A_3 M , A_2 N A_1 G R A_3 M , \ldots \)
\( \Delta = 3 \times 2 \times 1 \)
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\text{ANAGRAM} \\
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\[
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\[ \Rightarrow \frac{7!}{3!} \] Second rule!
Some Practice.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways for second, 1 for last.

\[ 3 \times 2 \times 1 = 3! \text{ orderings} \]

How many orderings of the letters in ANAGRAM?

Ordered, except for A!

Total orderings of 7 letters.

\[ 7! \]

Total "extra counts" or orderings of three A's?

\[ 3! \]

Total orderings?

\[ 7! \times \frac{1}{3!} \]

How many orderings of MISSISSIPPI?

4 S's, 4 I's, 2 P's.

11 letters total.

\[ 11! \times \frac{1}{4! \times 4! \times 2!} \]

= \[ 11! \times \frac{1}{4! \times 4! \times 2!} \]
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\( 11! \) ordered objects.

\( 4! \times 4! \times 2! \) ordered objects per “unordered object”
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$$\Rightarrow \frac{11!}{4!4!2!} \cdot$$
Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 \cdot \binom{52}{4} + \binom{52}{3}.$$
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.
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Wait a minute! Same as choosing 5 cards from 54 or \(\binom{54}{5}\).

Theorem:

\[
\binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}.
\]

Algebraic Proof:

Why? Just why? Especially on Thursday!

Already have a combinatorial proof.
Sum Rule

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No jokers "exclusive" or One Joker

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Theorem: \(\binom{54}{5}\)
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Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$. 
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In $T$. $\implies$ $|T|$
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\[
\begin{align*}
\text{In } T. & \implies |T| \\
\text{In } S. & \implies + |S|
\end{align*}
\]
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In $T$. $\implies |T|

In $S$. $\implies + |S|

Elements in $S \cap T$ are counted twice.
Simple Inclusion/Exclusion

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**Inclusion/Exclusion Rule:**
For any \( S \) and \( T \), \( |S \cup T| = |S| + |T| - |S \cap T| \).

![Diagram](image)

- \( \text{In } T. \implies |T| \)
- \( \text{In } S. \implies + |S| \)
- \( \text{Elements in } S \cap T \text{ are counted twice.} \)
- \( \text{Subtract. } \implies -|S \cap T| \)
Simple Inclusion/Exclusion

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Inclusion/Exclusion Rule:
For any \( S \) and \( T \), \(|S \cup T| = |S| + |T| - |S \cap T|\).

Elements in \( S \cap T \) are counted twice. Subtract. \( \implies -|S \cap T| \)

\(|S \cup T| = |S| + |T| - |S \cap T|\)
Using Inclusion/Exclusion.

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**Example:** How many 10-digit phone numbers have 7 as their first or second digit?
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**Example:** How many 10-digit phone numbers have 7 as their first or second digit? 

$S =$ phone numbers with 7 as first digit.
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**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$
Using Inclusion/Exclusion.

**Inclusion/Exclusion Rule:** For any $S$ and $T$,

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**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit.
Using Inclusion/Exclusion.

**Inclusion/Exclusion Rule:** For any $S$ and $T$,
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$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

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$S \cap T =$ phone numbers with 7 as first and second digit.
Inclusion/Exclusion Rule: For any $S$ and $T$,
\[ |S \cup T| = |S| + |T| - |S \cap T|. \]

Example: How many 10-digit phone numbers have 7 as their first or second digit?

$S = \text{phone numbers with 7 as first digit.} \ |S| = 10^9$

$T = \text{phone numbers with 7 as second digit.} \ |T| = 10^9.$

$S \cap T = \text{phone numbers with 7 as first and second digit.} \ |S \cap T| = 10^8.$
Using Inclusion/Exclusion.

**Inclusion/Exclusion Rule:** For any $S$ and $T$, 
$|S \cup T| = |S| + |T| - |S \cap T|$.

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T =$ phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$. 
Counting.

First Rule:
Make object out of sequence of choices.
$\ n_i$ - number of choices for choice $i$.
Number of objects:
Counting.

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Second Rule:
When order doesn’t matter, divide out orderings.
Counting.

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Sum Rule:
Size of union of disjoint sets is sum of sizes.
Counting.

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Number of objects: \( \prod_i n_i \).

Second Rule:
When order doesn’t matter, divide out orderings.

Sum Rule:
Size of union of disjoint sets is sum of sizes.

Inclusion/Exclusion:
Size of union of sets is sum of sizes 
\textbf{minus} the size of intersection.