| CS 70 | Discrete Mathematics and Probability Theory | |
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| Spring 2019 | Babak Ayazifar and Satish Rao | HW 11 |

Due: Friday, April 19, 2019 at 10 PM

## Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

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## 1 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let $G$ denote the numbers of girls that the Browns have. Let $C$ be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of $G$ and $C$. Fill in the table below.

| | $C=1$ | $C=2$ | $C=3$ |
|---|---|---|---|
| $G=0$ | | | |
| $G=1$ | | | |

(c) Use the joint distribution to compute the marginal distributions of $G$ and $C$ and confirm that the values are as you'd expect. Fill in the tables below.

| | |
|---|---|
| $\mathbb{P}(G=0)$ | |
| $\mathbb{P}(G=1)$ | |

| $\mathbb{P}(C=1)$ | $\mathbb{P}(C=2)$ | $\mathbb{P}(C=3)$ |
|---|---|---|
| | | |

(d) Are $G$ and $C$ independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

## 2 More Family Planning

(a) Suppose we have a random variable $N \sim Geom(1/3)$ representing the number of children of a randomly chosen family. Assume that within the family, children are equally likely to be boys and girls. Let $B$ be the number of boys and $G$ the number of girls in the family. What is the joint probability distribution of $B$, $G$?

(b) Given that we know there are 0 girls in the family, what is the most likely number of boys in the family?

(c) Now let $X$ and $Y$ be independent random variables representing the number of children in two independently, randomly chosen families. Suppose that $X \sim Geom(p)$ and $Y \sim Geom(q)$. Find $\mathbb{P}(X < Y)$, the probability that the number of children in the first family ($X$) is less than the number of children in the second family ($Y$). (You may use the convergence formula for a Geometric Series: $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ for $|r| < 1$)

(d) Show how you could obtain your answer from the previous part using an interpretation of the geometric distribution.

## 3 Combining Distributions

(a) Let $X \sim Pois(\lambda), Y \sim Pois(\mu)$ be independent. Prove that $X+Y \sim Pois(\lambda+\mu)$.

*Hint*: Recall the binomial theorem, which states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

(b) Let $X$ and $Y$ be defined as in the previous part. Prove that the distribution of $X$ conditional on $X+Y$ is a binomial distribution, e.g. that $X|X+Y$ is binomial. What are the parameters of the binomial distribution?

*Hint*: Your result from the previous part will be helpful.

## 4 Darts

Yiming is playing darts. Her aim follows an exponential distribution with parameter 1; that is, the probability density that the dart is $x$ distance from the center is $f_X(x) = \exp(-x)$. The board's radius is 4 units.

(a) What is the probability the dart will stay within the board?

(b) Say you know Yiming made it on the board. What is the probability she is within 1 unit from the center?

(c) If Yiming is within 1 unit from the center, she scores 4 points, if she is within 2 units, she scores 3, etc. In other words, Yiming scores $\lfloor 5-x \rfloor$, where $x$ is the distance from the center. (This implies that Yimin scores 0 points if she throws it off the board). What is Yiming's expected score after one throw?

## 5 Uniform Means

Let $X_1, X_2, \ldots, X_n$ be $n$ independent and identically distributed uniform random variables on the interval $[0,1]$ (where $n$ is a positive integer).

(a) Let $Y = \min\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Y)$. [*Hint*: Use the tail sum formula, which says the expected value of a nonnegative random variable is $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > x)\,\mathrm{d}x$. Note that we can use the tail sum formula since $Y \geq 0$.]

(b) Let $Z = \max\{X_1, X_2, \ldots, X_n\}$. Find $\mathbb{E}(Z)$. [*Hint*: Find the CDF.]

## 6 Moments of the Exponential Distribution

Let $X \sim \text{Exponential}(\lambda)$, where $\lambda > 0$. Show that for all positive integers $k$, $\mathbb{E}[X^k] = k!/\lambda^k$. [*Hint:* Integration by Parts.]