Due: Friday, April 12, 2019 at 10 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Binomial Variance

Throw n balls into m bins uniformly at random. For a specific ball i, what is the variance of the number of roommates it has (i.e. the number of other balls that it shares its bin with)?

2 Working with Distributions

1. For each of the following scenarios, find the distribution of each of the following random variables. That means, find the possible values that it can take on and their associated probabilities.
   (a) Five fair coins are flipped and the random variable Y is defined as the number of tails observed.
   (b) Two dice are rolled and the random variable Z is defined as the product of the two numbers rolled.
2. Suppose a fair six sided dice is rolled until a number smaller than 3 is observed. Let N be the total number of times the dice is rolled. Find $\mathbb{P}(N=k)$ for $k=1,2,3,\ldots$.
3. Now suppose two six-sided dice are rolled and the two numbers observed are defined as X and Y.
   (a) Calculate $\mathbb{P}(X>3 \mid Y=1)$.
   (b) Let $Z=X+Y$. What is the range of Z?
   (c) Calculate $\mathbb{P}(X=1 \mid Z<4)$.

3 Geometric and Poisson

Let X be geometric with parameter p, Y be Poisson with parameter λ, and $Z = \max(X,Y)$. Assume X and Y are independent. For each of the following parts, your final answers should not have summations.

(a) Compute $P(X > Y)$.

(b) Compute $P(Z \geq X)$.

(c) Compute $P(Z \leq Y)$.

4 Exploring the Geometric Distribution

(a) Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent. Find the distribution of $\min\{X,Y\}$ and justify your answer.

(b) Let X, Y be i.i.d. geometric random variables with parameter p. Let $U = \min\{X,Y\}$ and $V = \max\{X,Y\} - \min\{X,Y\}$. Compute the joint distribution of (U,V)

(c) Prove that U and V are independent.

5 Boutique Store

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) State the name and parameters of the distribution of Y and Z.

(c) Prove that Y and Z are independent.

6 Student Life

In an attempt to avoid having to do laundry often, Marcus comes up with a system. Every night, he designates one of his shirts as his dirtiest shirt. In the morning, he randomly picks one of his shirts to wear. If he picked the dirtiest one, he puts it in a dirty pile at the end of the day (a shirt in the dirty pile is not used again until it is cleaned). When Marcus puts his last shirt into the dirty pile, he finally does his laundry, and again designates one of his shirts as his dirtiest shirt (laundry isn't perfect) before going to bed. This process then repeats.

(a) If Marcus has n shirts, what is the expected number of days that transpire between laundry events? Your answer should be a function of n involving no summations.

(b) Say he gets even lazier, and instead of organizing his shirts in his dresser every night, he throws his shirts randomly onto one of n different locations in his room (one shirt per location), designates one of his shirts as his dirtiest shirt, and one location as the dirtiest location. In the morning, if he happens to pick the dirtiest shirt, *and* the dirtiest shirt was in the dirtiest location, then he puts the shirt into the dirty pile at the end of the day and does not use that location anymore (it is too dirty now). What is the expected number of days that transpire between laundry events now? Again, your answer should be a function of n involving no summations.