CS 70 Spring 2019 — Discrete Mathematics and Probability Theory — Satish Rao and Babak Ayazifar — HW 2

Due: Friday, 2/8, 10 PM

Sundry

Before you start your homework, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Induction on Reals

Induction is always done over objects like natural numbers, but in some cases we can leverage induction to prove things about real numbers (with the appropriate mapping). We will attempt to prove the following by leveraging induction and finding an appropriate mapping.

Bob the Bug is on a window extending to the ground, trying to escape Sally the Spider. Sally has built her web from the ground to 2 inches up the window. Every second, Bob jumps 1 inch vertically up the window, then loses grip and falls to half his vertical height.

Prove that no matter how high Bob starts up the window, he will always fall into Sally's net in a finite number of seconds.

2 Grid Induction

Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes). Every second he does one of the following (if possible):

(i) Walk one step down, to $(i, j-1)$.

(ii) Walk one step left, to $(i-1, j)$.

For example, if he is at $(5,0)$, his only option is to walk left to $(4,0)$; if Pacman is instead at $(3,2)$, he could walk either to $(2,2)$ or $(3,1)$.

Prove by induction that no matter how he walks, he will always reach $(0,0)$ in finite time. (*Hint*: Try starting Pacman at a few small points like $(2,1)$ and looking all the different paths he could take to reach $(0,0)$. Do you notice a pattern?)

3 Stable Marriage

Consider a set of four men and four women with the following preferences:

men	preferences	women	preferences
A	1>2>3>4	1	D>A>B>C
B	1>3>2>4	2	A>B>C>D
C	1>3>2>4	3	A>B>C>D
D	3>1>2>4	4	A>B>D>C

(a) Run on this instance the stable matching algorithm presented in class. Show each day of the algorithm, and give the resulting matching, expressed as $\{(M,W),\ldots\}$.

(b) Suppose we relax the rules for the men, so that each unpaired man proposes to the next woman on his list at a time of his choice (some men might procrastinate for several days, while others might propose and get rejected several times in a single day). Prove that this modification will not change what pairing the algorithm outputs.

4 The Better Stable Matching

In this problem we examine a simple way to *merge* two different solutions to a stable marriage problem. Let R, R' be two distinct stable matchings. Define the new matching $R \wedge R'$ as follows:

> For every man m, m's date in $R \wedge R'$ is whichever is better (according to m's preference list) of his dates in R and R'.

Also, we will say that a man/woman *prefers* a matching R to a matching R' if he/she prefers his/her date in R to his/her date in R'. We will use the following example:

men	preferences	women	preferences
A	1>2>3>4	1	D>C>B>A
B	2>1>4>3	2	C>D>A>B
C	3>4>1>2	3	B>A>D>C
D	4>3>2>1	4	A>B>D>C

(a) $R = \{(A,4),(B,3),(C,1),(D,2)\}$ and $R' = \{(A,3),(B,4),(C,2),(D,1)\}$ are stable matchings for the example given above. Calculate $R \wedge R'$ and show that it is also stable.

(b) Prove that, for any matchings R, R', no man prefers R or R' to $R \wedge R'$.

(c) Prove that, for any stable matchings R, R' where m and w are dates in R but not in R', one of the following holds:

- m prefers R to R' and w prefers R' to R; or
- m prefers R' to R and w prefers R to R'.

[*Hint*: Let M and W denote the sets of mens and women respectively that prefer R to R', and M' and W' the sets of men and women that prefer R' to R. Note that $|M|+|M'| = |W|+|W'|$. (Why is this?) Show that $|M| \leq |W'|$ and that $|M'| \leq |W|$. Deduce that $|M'| = |W|$ and $|M| = |W'|$. The claim should now follow quite easily.]

(You may assume this result in subsequent parts even if you don't prove it here.)

(d) Prove an interesting result: for any stable matchings R, R', (i) $R \wedge R'$ is a matching [*Hint*: use the results from (c)], and (ii) it is also stable.

5 Examples or It's Impossible

Determine if each of the situations below is possible with the traditional propose-and-reject algorithm. If so, give an example with at least 3 men and 3 women. Otherwise, give a brief proof as to why it's impossible.

(a) Every man gets his first choice.

(b) Every woman gets her first choice, even though her first choice does not prefer her the most.

(c) Every woman gets her last choice.

(d) Every man gets his last choice.

(e) A man who is second on every woman's list gets his last choice.

6 Short Answer: Graphs

(a) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph? (An expression that may contain n.)

(b) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

(c) True or False: For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the n-dimensional hypercube. Justify your answer.

(d) A complete graph with n vertices where n is an odd prime can have all its edges covered with x edge-disjoint Hamiltonian cycles (a Hamiltonian cycle is a cycle where each vertex appears exactly once). What is the number, x, of such cycles required to cover the a complete graph? (Answer should be an expression that depends on n.)

(e) Give a set of edge-disjoint Hamiltonian cycles that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5, where an edge is written as a pair of vertices from the set $\{0,1,2,3,4\}$ - e.g: $(0,1),(1,2)$.)